Abstract

Political decisions are frequently taken in assemblies by delegates who vote on behalf of their respective constituencies. Many such institutions assign different voting weights to the delegates (e.g., in the EU Council of Ministers), or equivalently involve delegations of different sizes who vote as a single block (e.g., US Electoral College), on the ground that constituencies differ in population size. While voting weights (or seats) proportional to population size seem the most straightforward allocation rule, Penrose (1946) argued that weights should be such that voting power of a constituency (as measured by the Penrose-Banzhaf index) is proportional to the square root of its population. This rule implements the “one person, one vote” principle in binomial choice environments; it equalizes the indirect influence of all citizens on collective decisions a priori. This paper shows that Penrose’s rule also applies in non-dichotomous median voter environments if a 50% threshold is used in the assembly and voters’ ideal policies are independently and identically distributed. It does not extend to supermajority rules. And interim heterogeneity between constituencies calls for proportionality just to population size.

Keywords: equal representation, collective choice, voting power, constitutional design
1 Introduction

A basic characteristic of today’s democracies is the use of political representatives who take decisions on behalf of the citizens. The indirect participation of the latter often has the following form: delegates are elected in disjoint constituencies (bottom tier); these cast block votes for their respective constituency in a council, electoral college, or other assembly (top tier). If constituencies are, e.g., for geographical, ethnic, or historical reasons not equally sized, a weighting scheme is applied to the votes at the top tier in order to reflect different bottom-tier population numbers. A prominent example for such a two-tier voting system is the Council of Ministers of the European Union, which consists of a single representative from each member state. In most respects equivalent examples, where multiple representatives of each constituency vote as a single block, are provided by the US Electoral College, which elects the President of the United States, or the German Bundesrat.

An important normative question arises for the design of such two-tier voting systems: which rule ought to define the weights of the representatives? The question has given rise to heated discussion, threats and bad blood at political summits, and received attention from a wider public in the wake of several EU Treaty reforms. This paper investigates the fair design of two-tier voting systems in an idealized political environment. We focus on the egalitarian principle of “one person, one vote”, which is generally considered to be at the heart of modern democratic constitutions. It strives to guarantee that collective decisions depend only on how many votes an alternative gets, not on whose votes these are. This anonymity principle is sometimes considered a minimum requirement for decision making procedures to be called democratic (e.g., Dahl 1956, p. 37). The US Supreme Court adopted it as its standard for evaluating legislative apportionment and has required “that each citizen have an equally effective voice in the election” (cf. Reynolds v. Sims, 377 U.S. 533, 1964, p. 565).

Ideally, an equally effective voice and equal influence on political outcomes are required for the entire process of voting and collective decision making (see Christiano 1996, pp. 232ff). Given that elections of delegates in a two-tier system are only a means to an end, suitable top-tier voting weights are integral parts of its egalitarian design. This paper identifies voting weight allocation rules which – at least from a constitutional a priori perspective – guarantee each voter an equal chance to indirectly determine the policy outcome. Expected utilitarian, egalitarian, and Rawlsian welfare, or the congruence of indirectly determined top-tier decisions with direct ones, are attractive alternative yardsticks for the evaluation of two-tier voting systems.

Presumably the most intuitive solution to the problem of designing a fair two-tier voting system and to implementing the “one person, one vote” principle seems to assign weights proportional to population sizes. Yet, a seminal result from the literature on voting power\(^1\) is that citizens’ indirect powers are equalized when representatives’ weights conform to Penrose’s (1946) square root rule. It requires the choice of weights such that the voting powers

\(^1\)See Felsenthal and Machover (1998) or Laruelle and Valenciano (2008b) for overviews.
of the representatives, as measured by the Penrose-Banzhaf index (Penrose 1946, Banzhaf 1965), are proportional to the square root of the respective population. Penrose’s square root rule played an important role in the public discussion of a draft of an EU constitution (see, for example, the open letter by Bilbao et al. 2004). It also featured prominently in an escalating controversy between, most visibly, Poland and Germany at an EU summit in June 2007. In the scientific literature, the rule has provided the benchmark for numerous applied studies (e.g., Felsenthal and Machover 2001, 2004; Leech 2002; Fidrmuc, Ginsburgh, and Weber 2009).

Penrose’s result rests on independent, in expectation equiprobable, and binary decisions by all voters. In the resulting binomial voting model, the probability of a tie in a constituency with $n_j$ voters can, with Stirling’s formula, be approximated by $\sqrt{2}/\sqrt{n_j\pi}$. As an individual voter is decisive in his or her constituency iff the election is otherwise tied, individual voting power at the bottom tier is a multiple of $1/\sqrt{n_j}$ a priori. A constituency’s voting power at the top tier is the probability that the policy outcome is decided by how it casts its voting weight. If the constituency’s power is made proportional to $\sqrt{n_j}$, then voters from large constituencies are compensated for their lower chance to determine their constituency vote. Hence, the probability for any given citizen to be indirectly decisive and to exert effective influence is equal, irrespective of the home constituency.

Good and Mayer (1975), Chamberlain and Rothschild (1981) and recently Kaniovski (2008) have demonstrated that Penrose’s rule fails to be a priori egalitarian, and needs to be modified substantially, if individual “yes” and “no” votes are not equiprobable or if they are dependent across voters (e.g., because these have aligned preferences). Empirical studies of two-party elections have actually failed to confirm the prediction for the average closeness of ballots which underlies the square root rule for US and UK data (see Gelman, Katz, and Tuerlinckx 2002 and Gelman, Katz, and Bafumi 2004). Elections in large constituencies turn out to be relatively closer than those in small constituencies, but by a smaller margin than required for the square root to be egalitarian. These findings cast doubts on both the robustness and relevance of the square root rule.²

However, investigating voting rules which maximize utilitarian welfare, Barberà and Jackson (2006) have found that the optimal weights vary with the square root of population if countries are modeled as consisting of many fixed-size blocks of individuals whose preferences are perfectly correlated within a block and independent across blocks. Beisbart, Bovens, and Hartmann (2005) have evaluated alternative decision rules for the European Union with respect to total expected utility if the utilities of the average individuals from each country are identically and independently distributed. Their results indicate that, while none of the considered decision rules unambiguously outperforms all others independently of parameter assumptions, square root weights (in combination with a 62% majority threshold) yield the highest expected utility in case that densities over the utilities of a proposal are symmetric and single-peaked and proposals tend not to improve upon the status quo. Similarly, Beisbart and Bovens (2007) have shown that an allocation of weights in proportion to the square root of population achieves both the ideal of maximal ex-

²Also see Felsenthal and Machover (1998, pp. 71f).
pected utility and the ideal of equal expected utilities across constituencies provided that no correlation exists between the utilities of individuals within the same constituency.

Moreover, weights proportional to the square root of population have been recommended from a majoritarian perspective. For instance, Felsenthal and Machover (1999) have shown that, in the binomial voting model, square root weights minimize what they have labeled the mean majority deficit. This refers to the expected value of the difference, if it is positive, between the number of supporters of the alternative that is not chosen at the top tier and the number of its opponents, who happen to obtain their favored but non-majoritarian option by the two-tier system. Closely related, Kirsch (2007) has found that square root weights minimize the difference between the margin of representatives accepting or rejecting a proposal and the size of the popular margin. Finally, Feix et al. (2008) report simulation results which suggest that the probability of non-majoritarian decisions originating from the two-tier voting process is minimized by the square root rule if each citizen votes independently of the others and chooses each of two alternatives with equal probability.

The quoted studies – whether they confirm, dispute, or qualify Penrose’s original conclusions – have in common that they consider binary decisions. They may be useful when the decisions of an assembly are indeed dichotomous. But even questions of war or peace can involve different shades of grey. Political decision making by, e.g., the EU Council frequently concerns the level of a tax or subsidy, the intensity of (de-)regulation, the speed of budget reform, the scale of military intervention, and so forth; it is much more rarely about having a tax, a subsidy, (de-)regulation of an industry, etc. per se.

To our knowledge, Maaser and Napel (2007) is so far the only analysis of two-tier voting systems which takes this into account, i.e., which considers two-tier choices from a scale, say, the unit interval [0, 1] rather than {0, 1}. We here follow up on that investigation. We consider egalitarian design of two-tier voting systems in case that alternatives are elements of a one-dimensional convex policy space, all voters have single-peaked preferences with random ideal points, and delegate preferences are perfectly aligned with the median voter in the respective constituency (i.e., we disregard political agency problems).

Maaser and Napel (2007) have demonstrated by Monte Carlo simulations that choosing weights proportional to the square root of population size happens to be the most egalitarian from a continuum of allocation rules for a 50%-majority threshold at the top tier. This was interpreted as the limit version of Penrose’s rule extending from binary to significantly richer policy spaces, from random “yes”-or-“no” votes to strategic interaction that produces the core of spatial voting games. We here provide an analytical proof for this finding.

3Some instances of this so-called referendum paradox (see, e.g., Nurmi 1998), which in the US occurred, e.g., in the 2000 Presidential Elections, are unavoidable.

4However, Laruelle and Valenciano (2008a) suggest a “neutral” top-tier voting rule when policy alternatives give rise to a Nash bargaining problem, and Le Breton, Montero, and Zaporozhets (2011) investigate fair voting weights in case of decisions on surplus division, i.e., a simplex of policy alternatives.

5The limit refers to situations when the Penrose-Banzhaf index becomes proportional to the vector of voting weights (see Lindner and Machover 2004).
In particular, we show that the probability of a given constituency being pivotal at the top tier is approximately proportional to the constituency’s weight and to the probability of the corresponding delegate ideal point being located near the common mean. The result is non-trivial: it quantifies the probability of a random variable to be the weighted median from a collection of independent but non-identically distributed random variables. The pursued strategy of proof suggests that, in case of finitely many constituencies, an elementary, purely weight-based approximation of equal representation can typically be improved if one adopts the exact Penrose rule (i.e., if one solves the so-called inverse problem of finding weights such that the resulting Penrose-Banzhaf index is proportional to the square root of population; see Leech 2003) or, alternatively, a square root rule based on the Shapley-Shubik index. These improvements, however, turn out to be relatively small for random configurations of 15 and more constituencies.

The new result, concerning the limit behavior of the ratio between the pivot probabilities of two constituencies, is unfortunately restricted to a 50%-majority threshold. We report extensive Monte Carlo simulations which show that the approximate optimality of weight-based square root rules fails to extend to supermajority voting: egalitarian allocations require a greater weight advantage for larger constituencies, the greater the threshold. The optimal exponent of an elementary power law, which equates a constituency’s weight with its population size raised to the power of \( \alpha \), rises non-linearly in the voting threshold. A non-linear relationship prevails even if more sophisticated rules, based on the Penrose-Banzhaf or Shapley-Shubik index, are considered. While equal influence can be achieved by one simple weight allocation rule in binary environments – regardless of the majority requirement – this is not the case in the median voter world. In fact, citizens’ indirect influence on policy outcomes becomes less egalitarian, the greater the decision quota \( q \), even when the respectively optimal exponent \( \alpha^*(q) \) is chosen. Because inequality of representation potentially jeopardizes the legitimacy of two-tier voting systems, this is bad news if, like in the EU Council, qualified majority requirements are the norm.

Our finding of a square root rule equalizing voter influence refers to a voting model that is very different from Penrose’s. However, at a qualitative level, they share the premise that all voters are treated as a priori identical and independent of each other, at least behind a constitutional “veil of ignorance”. The only exogenous asymmetry acknowledged in the two tier voting process are constituency sizes. Far-reaching symmetry assumptions can be justified by the “principle of insufficient” reason, and often are regarded as an essential ingredient of constitutional a priori analysis. Indeed, if one thinks of a voting weight allocation rule for, say, the US Electoral College, the German Bundesrat, or the EU Council, as being fixed for decades or centuries, then historical preference correlation and past voting behavior have small informational value. One may ignore such information even for short design horizons on normative grounds.

However, reasonable minds can disagree about what a posteriori information is admissible behind a constitutional “veil of ignorance”. In particular, even if one disregards historical preference information, past coalition structures, etc., it seems plausible to suppose – and account for – greater similarity of citizens within than between constituencies. Arguably, population size differences between constituencies have a reason in most cases.
Some similarity within (and dissimilarity between) constituencies – be it language, religion, affluence, geography, general political allegiance, etc. – is likely responsible for the partition of citizens into constituencies, to which a weight allocation rule shall be applied.

We try to capture this by investigating voters whose ideal points are ex ante identically distributed, but exhibit positive correlation within constituencies. If the resulting interim heterogeneity between constituencies is sufficiently big, then a linear rule emerges: Equal representation is (approximately) achieved by choosing weights such that each representative’s voting power, as measured by the Shapley-Shubik index, is proportional to the respective constituency’s population size. This applies even if a supermajority threshold is used at the top tier.

The remainder of the paper is organized as follows. Section 2 spells out our baseline model. The analysis of the limit case in which the number of considered constituencies grows without bound is discussed in Section 3, and then Section 4 reports on possible representation improvements in case of relatively small numbers of constituencies. Section 5 presents Monte Carlo simulations which show that the (approximate) optimality of square root rules is restricted to simple majority voting thresholds in the median voter world. Section 6 investigates the effect of interim heterogeneity on equality of representation, and Section 7 concludes.

2 Two-tier decision making

Consider a large population of voters. Let \( C^{(m)} = \{C_1, \ldots, C_m\} \) be a partition of the population into \( m \) constituencies, where each constituency \( C_j \) is supposed to have an odd number \( n_j = |C_j| > 0 \) of members. For any issue on which a collective decision may be taken, the set of alternatives constitutes a convex one-dimensional policy space \( X \subseteq \mathbb{R} \). The preferences of all citizens are assumed to be single-peaked, \(^6\) and the respective individual ideal point \( \nu_{ij} \in X \) for \( i = 1, \ldots, n_j \) and \( j = 1, \ldots, m \) is taken to be a random variable with a continuous density function \( f_{\nu_{ij}} \) and cumulative distribution function \( F_{\nu_{ij}} \). Our benchmark case is that all ideal points are identically and independently distributed (i.i.d), i.e., the population-wide ideal point vector \( (\nu_{11}, \ldots, \nu_{mm}) \) has a product distribution where all marginal density functions \( f_{\nu_{ij}} \) equal some \( f \). For simplicity, we assume that \( f \) is symmetric, and w.l.o.g. take the distribution’s mean to be zero. Moreover, we presume that \( f \) is bounded away from zero on a nonempty interval around the mean.

A collective policy \( x \in X \) is decided on by a council of representatives \( R^{(m)} \) which consists of one representative from each constituency. We assume that citizen \( i \) from constituency \( C_j \) has probability \( p_{ij} \) to determine the policy position \( \lambda_j \) adopted by his representative, i.e.,

\[
p_{ij} \equiv \Pr(\lambda_j = \nu_{ij})
\]

where \( \sum_{i \in C_j} p_{ij} = 1 \) for all \( j = 1, \ldots, m \). In general, numerous scenarios are compatible

\(^6\)As noted by Riker (1961, p. 908) this already imposes some form of homogeneity, as all individuals at least agree that an optimal choice for the issue at stake exists.
with this assumption. For example, some constituencies could be dictatorships, others
might be direct or parliamentary democracies, and still others could involve some form of
oligarchy.

In \( R^{(m)} \), the votes of the representatives are aggregated according to a **weighted voting
rule**: Each constituency \( C_j \) has voting weight \( w_j \geq 0 \), and any subset \( S \subseteq \{1, \ldots, m\} \) of
representatives which achieves a combined weight \( \sum_{j \in S} w_j \) above \( q^{(m)} \equiv 0.5 \sum_{j=1}^{m} w_j \), i.e.,
a **simple majority** of total weight, can pass a proposal to implement some policy \( x \in X \).

Let \( \cdot : m \) be the permutation which, for any realization of ideal points \( \lambda_1, \ldots, \lambda_m \) of the
\( m \) representatives, orders them from left to right, i.e., such that \( \lambda_{1;m} \leq \lambda_{2;m} \leq \ldots \leq \lambda_{m;m} \).

Then let the random variable \( P : m \) be defined by

\[
P : m \equiv \min \left\{ l \in \{1, \ldots, m\} : \sum_{k=1}^{l} w_{k:m} > q^{(m)} \right\}.
\]

Player \( P : m \)'s ideal point, \( \lambda_{P : m} \), is the unique policy that beats any alternative \( x \in X \)
in a pairwise council vote, i.e., constitutes the **core** of the voting game defined by weights \( w_1, \ldots, w_m \) and quota \( q^{(m)} \). We assume that the policy agreed by \( R^{(m)} \) indeed lies in
the core, i.e., it equals the ideal point of the **pivotal representative** \( P : m \), who will also be
referred to as the **weighted median** of \( R^{(m)} \). Equilibrium analysis of non-cooperative legislative
bargaining models which support policy outcomes inside or close to the core is provided, for instance, by Banks and Duggan (2000) and Cho and Duggan (2009).

### 2.1 Median voter model

While the determination of constituency \( C_j \)'s representative might, in principle, take many
distinct forms, we are here interested in situations in which the representative’s ideal point
\( \lambda_j \) coincides with \( C_j \)'s **median voter**. Namely, we consider the case

\[
\lambda_j \equiv \nu_{j}^{\frac{n_j+1}{2}}:n_j
\]

where \( \nu_{j}^{k:n_j} \) refers to the \( k \)-th order statistic of random vector \( \nu_j = (\nu_{j1}, \ldots, \nu_{jn_j}) \).

Thinking of some form of political competition giving rise to this, or bargaining within each
constituency, we will proceed on the assumption that constituency \( C_j \)'s representative is
fully responsive to the constituency’s median voter. In particular, a sufficiently small shift
of \( \nu_{j}^{k:n_j} \) would translate into an identical shift of \( \lambda_j \) – and, in case that representative
\( j \) is pivotal at the top tier, i.e., \( P : m = j \), a corresponding shift of the collective decision.

This event corresponds to voter \( n_j \cdot \frac{j+1}{2} : n_j \) determining the policy outcome, i.e., this voter
has influence on the collective decision. Given that all ideal points \( \nu_j \) are assumed to be
i.i.d., the median voter assumption entails that each voter in constituency \( C_j \) has the same

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7We will disregard the zero probability event of two or more constituencies having identical ideal points.

8Recall that \( n_j \) is, for simplicity, assumed to be odd.
probability

\[ p_j^i = \frac{1}{n_j} \]

to determine \( \lambda_j \).\(^9\) Note that the distribution of \( \lambda_j \) differs from \( f \) (in contrast, for instance, to random dictatorship).

The actual distribution of \( \lambda_j \), and its density function \( f_{\lambda_j} \), can be calculated from \( F \). For instance, if each \( \nu_j^i \) is uniformly distributed on \([−\frac{1}{2}, \frac{1}{2}]\), then \( \lambda_j \) has a beta distribution with parameters \( \frac{n_j^i+1}{2} \) and \( \frac{n_j^i+1}{2} \) (shifted by \( \frac{1}{2} \)). But we will focus on reasonably large constituency sizes \( n_j \), so that details of the underlying individual ideal point distribution do not matter. Namely, regardless of \( F \), the median position \( \lambda_j \) in constituency \( C_j \) is asymptotically normally distributed (see, e.g., Arnold et al. 1992, p. 223) with mean

\[ \mu_j = F^{-1}(0.5) \]

and standard deviation

\[ \sigma_j = \frac{1}{2f(F^{-1}(0.5))\sqrt{n_j}}. \]  \(1\)

The larger a constituency \( C_j \) is, the more concentrated is the distribution of its median voter’s ideal point, \( \lambda_j \), around the median of the underlying distribution \( F \) of individual ideal points (here: \( \mu = 0 \)). In particular, \( n_j > n_k \) implies that the median distribution \( F_{\lambda_k} \) is a mean-preserving spread of \( F_{\lambda_j} \).

Of course, the considered median voter environment is a big simplification. Many collective decisions involve more than just a single dimension in which voter preferences differ. Even if the assumption of a one-dimensional policy space and single-peaked preferences is granted, systematic abstention of certain social groups can drive a wedge between the median voter’s and the median citizen’s preference; then equal influence of voters and non-voters cannot be achieved for any voting weight allocation rule.\(^10\) Moreover, we ignore the important issue of agency problems, connected to imperfect monitoring and usually rather infrequent delegate selections (for instance, national elections in the EU member states usually take place every four to five years). Empirical evidence highlights that a representative may take positions that differ significantly from his district’s median when voter preferences within that district are sufficiently heterogeneous (see, e.g., Gerber and Lewis 2004). But intuitions about fairness commonly relate to abstract thought experiments of a “veil of ignorance” kind. Our analysis of a stylized decision framework – a frictionless world with particularly well-behaved, a priori perfectly symmetric preferences – is meant to provide a transparent benchmark.

\(^9\)If voters within a constituency cannot be considered i.i.d. at least a priori, the normative foundation for trying to give each equal a priori power on the collective decision seems weak.

\(^10\)A related issue are members without suffrage (e.g., minors, immigrants, or prisoners). However, the political discussion of, say, the voting weights in the EU Council has almost exclusively referred to population figures, rather than numbers of eligible or actual voters. We here follow this line, i.e., refer to citizens and voters interchangeably.
3 Egalitarian Voting Weights

By egalitarian representation in a two-tier voting system we mean that each voter in every constituency has equal probability to be critical for the collective decision, i.e., to indirectly determine the policy outcome selected by weighted voting of representatives at the top tier.\footnote{11} In above notation, this amounts to requiring that for some constant \( c > 0 \)

\[
\Pr (j = P : m \cap \lambda_j = \nu_j^i ) = p_j^i \cdot \Pr (j = P : m \mid \lambda_j = \nu_j^i ) \equiv c,
\]

for all \( j \in \{1, \ldots , m\} \) and \( i \in C_j \). We call a weight assignment \( w_1, \ldots , w_m \) egalitarian if it satisfies condition (2) for the voting threshold \( \sum_j 0.5w_j \). Note that the assumption of \( \sum_{i \in C_j} p_j^i = 1 \) for all \( j = 1, \ldots , m \) imposes \( c = \frac{1}{\sum_{j=1}^m n_j} \).

The combinatoric properties of voting imply that perfectly egalitarian representation by delegates may be impossible to achieve by any deterministic voting rule.\footnote{12} For illustration, consider two equally sized constituencies, \( A \) and \( B \), and a third one, \( C \), which comprises five times more voters. Weights that render \( C \) either a dictator or symmetric to \( A \) and \( B \) must violate (2); weights that make \( \{A, C\} \) and \( \{B, C\} \) but not \( \{A, B\} \) winning coalitions satisfy (2) only for distributions of \( \lambda_A \), \( \lambda_B \), and \( \lambda_C \) which make \( A \) or \( B \) the unweighted median with a knife-edge probability of \( \frac{1}{4} \). We, therefore, generally have to be satisfied with representation which is \textit{approximately egalitarian}, meaning that each term \( p_j^i \cdot \Pr (j = P : m \mid \lambda_j = \nu_j^i ) \) is close to being proportional to \( \frac{1}{\sum_{j=1}^m n_j} \).

In general, equal representation cannot be achieved if \( p_j^i = 0 \) for some individual \( i \) in some constituency \( C_j \). Provided that all individuals have strictly positive probability \( p_j^i > 0 \) of determining the outcome in their constituency, representation may happen to be equal even if \( p_j^i \neq p_h^i \) for two individuals \( i, h \in C_j \) in some constituency \( C_j \). Namely, the events \( \{\lambda_j = \nu_j^i\} \) and \( \{j = P : m\} \) need not be independent. A smaller probability of voter \( h \) determining constituency \( C_j \)'s position might be balanced by a greater conditional probability of \( C_j \) being the weighted median at the top tier in case of \( \lambda_j = \nu_j^h \).

This, however, is ruled out if all voter ideal points \( \nu_j^i \) are i.i.d. and \( \lambda_j \) corresponds to \( C_j \)'s median. In this model,

\[
\Pr (j = P : m \mid \lambda_j = \nu_j^i ) = p_j^i \cdot \Pr (j = P : m),
\]

and any voter \( i \) in constituency \( C_j \) determines its representative’s policy position with probability \( p_j^i = \frac{1}{n_j} \). Therefore, if we denote representative \( j \)'s probability to be pivotal in council \( \mathcal{R}^{(m)} \) by

\[
\pi_j (\mathcal{R}^{(m)}) \equiv \Pr (j = P : m)
\]

\footnote{11} An alternative egalitarian criterion – namely, seeking to equalize the expected utility of voters under the assumption of an interpersonally comparable cardinal utility function – is briefly considered in Appendix A. A heuristic argument shows that a square root rule seems to be egalitarian also with respect to voter utility, even if higher egalitarian utility is likely reached if delegates represent their mean rather than median voter.

\footnote{12} If \( p_j^i = p_j > 0 \) for all \( i \in C_j \), it could be achieved by a probabilistic rule, such as making representative \( j \) the top-tier dictator with probability \( \frac{n}{\sum_{k=1}^m n_k} \).
then equal representation condition (2) can be rewritten as the requirement that

\[ \frac{\pi_j(R^{(m)})}{\pi_k(R^{(m)})} = \frac{n_j}{n_k} \]  

for any \( j, k \in \{1, \ldots, m\} \). If constituency \( C_j \) is twice as large as constituency \( C_k \) then representative \( j \) must have twice the chance to be pivotal than representative \( k \) in order to equalize the probability for any individual voter to be decisive for the collective outcome.

In the baseline model, distributions \( F_{\lambda_j} \) and \( F_{\lambda_k} \) are related by second-order stochastic dominance; the representatives (i.e., medians) from large constituencies have greater probability to be the unweighted median in \( R^{(m)} \) than smaller ones. Linear weights, e.g., \( w_j = n_j \), would therefore tend to result in under-representation of small constituencies. In particular, the probability of the event \( \{j = P : m\} \) would equal the Shapley-Shubik index (SSI) of representative \( j \) if all ideal points \( \lambda_l \) were i.i.d. In that case – acknowledging that a voter’s SSI is close to its relative weight for large \( m \) except for pathological cases (see Lindner and Machover 2004) – a linear rule would give the representative from the smaller constituency, say \( C_k \), approximately the pivot probability \( \frac{1}{n_k} \) needed to equalize indirect influence. However, the actual probability \( \pi_k(R^{(m)}) \) is smaller because the more spread-out distribution of \( \lambda_k \) has less mass in the relevant range of \( X \) than in case of all \( \lambda_l \) being i.i.d. So small constituencies would have a disadvantage. Conversely, under constant weight, say, \( w_j = 1 \) for all \( j \), the more concentrated distribution of \( \lambda_j \) compensates the citizens from \( C_j \) only partially for having a smaller probability \( \frac{1}{n_j} \) to be critical within their constituency.

**Proposition 1.** Consider a chain \( R^{(1)} \subset R^{(2)} \subset R^{(3)} \subset \ldots \) of councils for which the voting weights only assume a finite number of positive integers \( \tilde{w}_1, \tilde{w}_2, \ldots, \tilde{w}_r \). Let each representative \( j \) with \( w_j = \tilde{w}_s \) have an ideal point \( \lambda_j \) with \( E[\lambda_j] = 0 \) that is distributed according to the symmetric density \( f_{\lambda_j} = \tilde{f}_s \), where \( \tilde{f}_s \) is continuous on \([-\varepsilon_1, \varepsilon_1] \) for \( \varepsilon_1 > 0 \) and satisfies \( \tilde{f}_s(0) > 0 \). Moreover, for each \( s \in \{1, \ldots, r\} \), let \( \lim_{m \to \infty} |\{k \in \{1, \ldots, m\} : w_k = \tilde{w}_s\}|/m = \beta_s > 0 \) with \( \sum_{s=1}^r \beta_s = 1 \). Then

\[ \lim_{m \to \infty} \pi_i(R^{(m)}) = \frac{w_i f_{\lambda_i}(0)}{w_j f_{\lambda_j}(0)}. \]  

(4)

The proof is presented in Appendix B. A direct corollary of (1), this result, and the fact that the pivot probabilities of all constituencies sum to one is

**Corollary 1.** Under the conditions of Proposition 1 and assuming that the ideal point of any representative \( j \in \{1, \ldots, m\} \) coincides with the median of \( n_j \) i.i.d. ideal points,

\[ \pi_i(R^{(m)}) \approx \frac{w_i \sqrt{n_i}}{\sum_{k=1}^m w_k \sqrt{n_k}} \]  

for large \( m \).
The key observation behind limit (4) and its corollary is that, as \( m \) grows large, the pivotal member of \( \mathcal{R}^{(m)} \) is most likely found very close to the common mean and median of the ideal point distributions \( F_{\lambda_1}, \ldots, F_{\lambda_m} \). The reason is that pivotality at location \( x \in X \) requires that exactly half the total weight of \( \mathcal{R}^{(m)} \)'s members is "located" in \( (-\infty, x] \), i.e., the aggregate relative weight of representatives with ideal points \( \lambda_j \in (-\infty, x] \) must equal \( \frac{1}{2} \). In expected terms, this occurs exactly at \( x = 0 \). The probability for the realized weighted median in \( \mathcal{R}^{(m)} \) to fall outside an \( \varepsilon \)-neighborhood of 0 approaches zero exponentially fast as \( m \to \infty \), while individual pivot probabilities vanish much more slowly.

One can, therefore, restrict attention to a sufficiently small interval \( (-\varepsilon, \varepsilon) \) for large \( m \). The densities \( f_{\lambda_1}(x), \ldots, f_{\lambda_m}(x) \) are approximately constant on this interval by continuity. Moreover, conditioning on the event \( \{\lambda_j \in (-\varepsilon, \varepsilon)\} \), they are almost identical for any \( j = 1, \ldots, m \). This makes all orderings of the representatives with ideal points in \( (-\varepsilon, \varepsilon) \) approximately equiprobable. Therefore, these representatives’ respective conditional pivot probabilities correspond to their Shapley-Shubik index, which is known to become proportional to the respective weight vector in the limit. It then remains to acknowledge that the probability of the condition \( \{\lambda_j \in (-\varepsilon, \varepsilon)\} \) being true is proportional to \( \lambda_j \)’s standard deviation \( \sigma_j \), which is a linear function of \( \sqrt{n_j} \) if representatives correspond to the median of \( n_j \) i.i.d. draws from some density \( f \). One may even directly think of the limit case \( \varepsilon \to 0 \): if one conditions on representative \( j \)'s ideal point being located at \( x = 0 \), i.e., \( \{\lambda_j = 0\} \), then each representative \( k \neq j \) is equally likely found to \( j \)'s left or right (because \( F_{\lambda_k}(0) = \frac{1}{2} \)). In this case, \( j \)'s conditional pivot probability equals \( j \)'s Penrose-Banzhaf index, which, like the Shapley-Shubik index, becomes proportional to \( w_j \) in the limit, with very few exceptions (see Lindner and Machover 2004 and Chang, Chua, and Machover 2006). The density of the condition \( \{\lambda_j = 0\} \) is, as already noted, proportional to \( \sqrt{n_j} \) for the median of \( n_j \) i.i.d. draws.

Combining (3) and (4), it is straightforward to conclude

**Proposition 2.** Under the conditions of Proposition 1 and assuming that the ideal point of any representative \( j \in \{1, \ldots, m\} \) coincides with the median of \( n_j \) i.i.d. ideal points, representation is approximately egalitarian for large \( m \) if weights \( w_1, \ldots, w_m \) are chosen proportional to \( \sqrt{n_1}, \ldots, \sqrt{n_m} \).

So, except in a pathological cases which are avoided by imposing the sufficient conditions of Proposition 1, egalitarian representation is approximated in the i.i.d. median voter setting by a simple square root rule.

4 **Sophisticated square root rules**

The number of constituencies \( m \) need not be very large for Proposition 2 to apply. This has been demonstrated in the simulation study of Maaser and Napel (2007). It focused on the investigation of elementary power laws

\[ w_j = n_j^\alpha \quad (5) \]
with \( \alpha \in [0, 1] \). Approximately, for large \( m \), this includes Penrose’s square root rule as the special case \( \alpha = 0.5 \). The measure that was used to evaluate the performance of weight allocation rules was the *cumulative quadratic deviation* between the realized and the ideal probabilities of any individual being doubly pivotal, namely

\[
\sum_{j=1}^{m} n_j \cdot \left( \frac{1}{\sum_{k=1}^{m} n_k} - \frac{\hat{\pi}_j}{n_j} \right)^2.
\]

(6)

Here \( \hat{\pi}_j \) refers to the estimate of \( \pi_j(\mathcal{R}^{(m)}) \) obtained by averaging over a large number of draws from \( F_{\lambda_1}, \ldots, F_{\lambda_m} \). This criterion weights deviations at the level of individuals equally, which makes egalitarian values \( \pi_j(\mathcal{R}^{(m)}) \) of very populous constituencies \( C_j \) particularly important. The elementary square root rule formulated in Proposition 2, which corresponds to \( \alpha = 0.5 \), turned out to be the most egalitarian (and close to being perfectly so) for a large number of randomly generated population configurations and some real world examples.

Usually, this applies already for \( m \geq 15 \). However, convergence can be slow, and requires \( m \geq 50 \), if artificial configurations are considered which involve population sizes that have small variance or are extremely skewed. This highlights that the combinatoric properties of weighted voting cannot be disregarded altogether. After all, these have been the reason for the development of voting power indices by, amongst others, Penrose (1946), Shapley and Shubik (1954), and Banzhaf (1965). They, and many related indices, are mappings from *simple games* with a characteristic function \( v: 2^N \to \{0, 1\} \), that describes which subsets \( S \subseteq N \) of a given set of players \( N \) can pass a proposal, to real vectors whose \( i \)-th component summarizes player \( i \)'s possibilities to be decisive in \( v \), i.e., to form and be critical for a (minimal) winning coalition. Weighted voting games constitute a proper subset of the space of simple games (see Taylor and Zwicker 1999).

Even though the framework considered in this paper is very different from the one underlying conventional power indices, the indicated intuition for Propositions 1 and 2 suggests that they may nevertheless be useful when the number of constituencies, \( m \), is insufficiently great. The reason is that, at least for a particular distribution over all coalitions \( S \subseteq N \) (e.g., uniform), they explicitly account for the combinatoric properties of weighted voting. This section, therefore, conducts a short simulation study in order to investigate the extent to which “sophisticated square root rules”, which are based on either the Shapley-Shubik index (SSI) or the Penrose-Banzhaf index (PBI), improve equality of representation as measured by (6) relative to Proposition 2.

Unfortunately, implementing such sophisticated rules requires solutions to the so-called *inverse problem* of finding voting weights which induce a desired power index vector. For a finite number \( m \) of council members, the number of different voting rules increases very quickly in \( m \), but remains finite. Therefore, the set of inducible power vectors is discrete, as illustrated in Figure 1 for \( m = 3 \). Moreover, the problem of enumerating all simple games with \( m \) players (thus, e.g., all inducible Shapley-Shubik power vectors) is very hard. It could, in principle, be solved by determining all *antichains* on \( 2^N \).\textsuperscript{13} But because even

\textsuperscript{13}A subset of a partially ordered set \( (P, <_P) \) – where \( P \) is a (finite) set and \( <_P \) is a partial order of
finding the number of different antichains on \(2^N\) is an unsolved problem (also known as Dedekind’s problem), we consider a numerical algorithm for solving the inverse problem approximately – similar to, e.g., Leech (2003) and Leech and Machover (2003).

For the comparison, we use 30 randomly generated configurations of \(m = 15\) constituencies each. In half of the configurations, population sizes are drawn from a uniform distribution at the start of the simulation, and in the other half from a Pareto distribution with skewness parameter \(\kappa = 1.0\) (the latter matches real population distributions better). Experience suggests that \(m = 15\) is large enough to avoid extreme discrepancies between voting weights and voting power, which can be caused by the combinatorial particularities of the configuration at hand,\(^\text{14}\) while asymptotic behavior only begins to kick in (cf. Chang, Chua, and Machover 2006). For larger numbers of constituencies, it becomes increasingly difficult to discriminate meaningfully between elementary weight-based and sophisticated index-based rules. The reason is that the power ratios between any two representatives, as measured by their SSI or PBI, typically approaches the ratio of their voting weights for simple majority thresholds (see Lindner and Machover 2004, and Lindner and Owen 2007).

Let \(w^{SSI} = (w_1^{SSI}, \ldots, w_m^{SSI})\) refer to the (numerical) solution vector to the problem of finding weights for representatives \(j = 1, \ldots, m\) such that their induced Shapley-Shubik index values, \(SSI_j\), are approximately proportional to \(\sqrt{n_j}\). Analogously, let \(w^{PBI}\) denote the solution to the inverse problem involving the Penrose-Banzhaf index and square root of population sizes. Figure 2 compares the cumulative individual quadratic deviations from egalitarian representation (see (6)) which is obtained for elementary square root

\(^{\text{14}}\)Examples are, for instance, the SSI or PBI vectors of \((1,0,0)\) or \((\frac{1}{3}, \frac{1}{3}, \frac{1}{3})\) for the 3-player weight distributions \(w = (51\%, 47\%, 2\%)\) or \(w' = (49\%, 49\%, 2\%)\) under simple majority.
Figure 2: Cumulative individual quadratic deviation under $w_j = \sqrt{n_j}$, $w^{PBI}$, $w^{SSI}$, and $w^*$
weights \( w_j = \sqrt{n_j} \), \( j = 1, \ldots, m \) ("simple weights"), for \( w_{SSI} \) ("SSI inverse") and for \( w_{PBI} \) ("PBI inverse"). Moreover, as a benchmark for the weight allocations obtained by these three different square root rules, the figure also shows the corresponding deviation under configuration-specific, most egalitarian weights \( w^* \) ("best unconstrained"). They are obtained from an unconstrained minimization of objective function (6).

Panel (a) shows results for 15 distinct configurations with uniformly distributed constituency sizes, and panel (b) the same number of configurations with Pareto distributed constituency sizes. The cumulative deviation which is associated with the respective weight vector \( w^* \) can be considered inevitable: it owes to the discrete nature of the set of possible power allocations and, in general, cannot be eliminated completely. Though not very clearly, Figure 2 suggests that systematic differences in the performance of the four distinct sets of weights may exist. The graphic impression is corroborated by a statistical comparison (pooling all 30 observations) of cumulative individual quadratic deviations for [1a] \( w_{PBI} \) vs. \( w_j = \sqrt{n_j} \) [1b] \( w_{SSI} \) vs. \( w_j = \sqrt{n_j} \), [2a] \( w_{PBI} \) vs. \( w^* \), and [2b] \( w_{SSI} \) vs. \( w^* \), and [3] \( w_{PBI} \) versus \( w_{SSI} \). The Wilcoxon signed rank test (see, e.g., Hollander and Wolfe 1999, Ch. 3) rejects the null hypothesis that the median difference between pairs of observations is zero at the 99% significance level in comparisons [1a] and [1b]. So both the inverse weights \( w_{PBI} \) and \( w_{SSI} \) perform significantly better than the elementary rule espoused by Proposition 2. Similarly, the null hypotheses in tests [2a] and [2b] are rejected at the 99% significance level. So neither \( w_{PBI} \) nor \( w_{SSI} \) represent first-best solutions in the considered median voter environment. This is no surprise; it highlights that the objective of finding a general rule that approximately solves the problem of egalitarian representation for a large class of constituency configurations (rather than specifically for, say, the EU of currently 27 members and 2011 population data), comes at a (small) price. Finally, in test [3], the null hypothesis could not be rejected. So no significant difference between equality of representation under \( w_{PBI} \) and \( w_{SSI} \) could be detected.

These differences should become negligible for larger numbers of constituencies. To confirm this, an additional test including 12 configurations with \( m = 30 \) constituencies each has been conducted. For these, cumulative deviations under weights \( w_{SSI} \) have been compared with those for \( w_j = \sqrt{n_j} \) and also the configuration-specific best power law weights \( w_j = n_j^{\alpha^*} \). The null hypotheses that the median difference between pairs of observations is zero could not be rejected at the 95% significance level (sometimes at the 90% level).

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15Here, “unconstrained” means that there are \( m - 1 \) degrees of freedom, rather than a single one as in rules of the form \( w_j = n_j^\alpha \), or sophisticated ones referring to some index. The respective vector \( w^* \), as well as \( w_{PBI} \) and \( w_{SSI} \), are obtained numerically by the Nelder-Mead simplex method (see, for example, Avriel 1976, Ch. 9). In each step of the search, the probabilities \( \pi_j(R^{(m)}) \) are approximated by empirical frequencies of event \( \{ P; m = j \} \) in \( 10^7 \) iterations. The MATLAB source code is available upon request.

16Generally, \( \alpha = 0.5 \) is not exactly the best exponent among all power laws for \( m < \infty \). However, even when the configuration-specific best power law \( w_j = n_j^{\alpha^*} \) is considered, \( w_{PBI} \) and \( w_{SSI} \) perform better.

17The 12 configurations consisted of \( 3 \times 4 \) configurations with population sizes drawn from a uniform, a normal, and a Pareto (\( \kappa = 1.0 \)) distribution, respectively.
5 Supermajority Rules

Voting thresholds or quotas which exceed 50% of total weight, i.e., supermajority voting rules, play an important role in theory (see, e.g., Buchanan and Tullock 1962; Caplin and Nalebuff 1988) and also in practice. In particular, the decision rules for the EU Council have consistently involved so-called qualified majority requirements. With applications in the EU context in mind, but also in general, it is worthwhile to investigate the robustness of the square root rule in the median voter environment. We will therefore adapt our definition of the pivotal member $P:m$ in a top-tier council or electoral college $R^{(m)}$, and allow quotas exceeding 50%.

A voting game with supermajority rule is, in general, not strong, i.e., the complement of a losing coalition (one without sufficient weight) need not be winning. This implies that no policy $x \in X$ need exist which defeats all alternatives $x' \neq x$ in pairwise comparison. It may be impossible to bring about the necessary majority for any change of a given legislative status quo. The probability that collective decision making merely confirms the status quo is a measure of the institutional inertia created by the decision threshold.

The following analysis, however, concentrates on creative power rather than representatives’ abilities to preserve the status quo. To this end, a status-quo policy, $Q$, is fixed to a point outside $R^{(m)}$’s Pareto set $[\lambda_{1:m}, \lambda_{m:m}]$. In other words, this section will assume that all members of $R^{(m)}$ agree on the direction of desirable policy change. For specificity, let $Q \leq \lambda_{1:m}$. This implies that some winning coalition will, in a frictionless world, displace $Q$ by some policy to its right. This would be the case even if unanimity rule were used. Just for illustration, suppose $Q = 0$ and let a continuum of representatives have equal weights and uniformly distributed ideal points on $X = [0, 1]$. Then, for any given value of a relative decision quota $q \in (0.5, 1)$, all policies $x \in (0, 2(1 - q))$ are preferred to the status quo by a majority of at least measure $q$. Any policy $x < 1 - q$ could still be improved upon by a share of representatives greater than $q$. A continuous process of displacements of the momentary status quo can be expected to come to a halt at $x = 1 - q$. Any suggested further movement to the right will be blocked by the representative whose ideal point equals $1 - q$, and those to his left; supporters of the move do not wield the necessary weight any longer. The policy outcome under a quota of, e.g., $q = 0.75$ would be the first quartile point of the distribution of representative ideal points.

Considering weighted voting and a finite number of representatives, analogous reasoning suggests that the policy adopted by $R^{(m)}$ coincides with the ideal point of the representative who is pivotal “from the right”. This amounts to re-defining

$$P_q \equiv \min \left\{ l \in \{1, \ldots, m\} : \sum_{k=1}^{l} w_{k:m} > (1 - q) \sum_{j=1}^{m} w_j \right\}$$

for a given relative decision quota $q \in (0.5, 1)$.

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18 One could equivalently consider $Q$ to be random, condition on events in which a supermajority is in favor of displacing $Q$, and then consider the member of $R^{(m)}$, who is the least enthusiastic supporter of change, to be the pivotal one.
In principle, it might be feasible to find analytical arguments in order to obtain approximations of the implied pivot probabilities \( \pi_j(q; R(m)) \equiv \Pr(j = P_q; m) \) under definition (7). But at least our analytical arguments leading to Proposition 1 cannot straightforwardly be extended to \( q > 0.5 \) (except for distributions \( F_{\lambda_1}, \ldots, F_{\lambda_m} \) which have identical \( q \)-quantiles). We, therefore, resort to Monte Carlo simulations and compare rules of the form \( w_j = n_j^\alpha \). This directly extends the investigation of Maaser and Napel (2007) to a range of alternative quotas.\(^{19}\) Again, the extent to which the considered rule falls short of the egalitarian ideal (see (3)) will be measured by the cumulative quadratic deviation at the individual level, as given by (6). First, a number of randomly generated configurations will be investigated; then the EU Council is briefly considered as a real-world example.

Note already that even approximately equal representation is not achievable for sufficiently distinct population sizes under unanimity rule. This suggests that the “ideal” voting weight assignment must produce greater deviations from the egalitarian benchmark for \( q \gg 0.5 \) than for \( q = 0.5 \). Moreover, the benchmark of unanimity rule, which defines a symmetric weighted voting game, highlights that a fixed weight advantage of some constituency \( C_j \) over \( C_k \) tends to matter less, the greater the quota. In order to maintain the same advantage in voting power of a more populous \( C_j \), which is required to equalize the expected indirect influence of individual voters, a greater weight advantage (i.e., greater \( \alpha \)) is likely to be needed.

### 5.1 Randomly generated configurations

Tables 1, 2, and 3 report optimal values of \( \alpha \) for four random configurations with \( m = 30 \) constituencies each. They consider different probability distributions of the constituency sizes. As mentioned, the difference between simple and sophisticated rules – square root or other – becomes less and less significant for \( m \gg 15 \). We therefore focus on the computationally less demanding rules of the elementary type \( w_j = n_j^\alpha \), where \( \alpha \in [0,1] \) is increased in steps of 0.01. The respective probabilities \( \pi_j(q; R(m)) \) are estimated by simulations with \( 10^7 \) iterations. The cumulative deviations associated with the respective optimal value of \( \alpha \) are shown in the tables in parentheses.

Three observations apply irrespectively of the considered distribution of constituency sizes. First, the most egalitarian value \( \alpha^*(q) \) increases in the quota \( q \). There is an effect that adds to weight advantages generally having smaller ramifications, the closer we approach unanimity rule. Namely, the median voter of large constituencies is more concentrated around the common median of distributions \( F_{\lambda_1}, \ldots, F_{\lambda_m} \). It is relatively rarely found near the location \( x \) where a supermajority \( q \gg 0.5 \) is accumulated by representatives with ideal points \( \lambda_j \in (-\infty, x) \) (or, considering a status quo on the very left, those with \( \lambda_j \in (x, \infty) \)). This implies a smaller probability of being pivotal, unless the voting weights of the more populous countries are increased relative to the square root baseline. Second, as anticipated, the deviation from perfectly egalitarian pivot probabilities indeed increases

\(^{19}\)An additional benefit of the extensive simulations reported in this section is their increased precision relative to Maaser and Napel (2007).
in $q$. From $q = 55\%$ to $q = 80\%$, the quality of representation deteriorates by a factor of up to 1000. This decline is mainly caused by discrete weight vectors either producing too little or too much probability for the large constituencies. And, third, while one might have expected the cumulative individual quadratic deviations to be lowest under simple majority, they actually reach their minimum at a quota of 55% (among the quotas considered here). We do not have a clear intuition for this.

Table 1 shows results for uniformly distributed constituency sizes $n_1, \ldots, n_{30}$. Populations in configurations (I) and (II) come from a uniform distribution on $[0, 10^8]$; those in (III) and (IV) from one on $[3 \cdot 10^6, 10^7]$. It is readily noticed that the deviations (in parentheses) are smaller in columns (I) and (II) than deviations in (III) and (IV), except for the highest quotas. There, no systematic difference is apparent. Moreover, the respective optimal $\alpha$ varies less between configuration (I) and (II) than between (III) and (IV).

As the variance of $U(0, 10^8)$ is by a factor of 200 greater than that of $U(3 \cdot 10^6, 10^7)$, the results vaguely indicate a positive relationship between population size variance and the robustness of the optimal equal representation rule.

This relationship is corroborated by the data for normally distributed populations in Table 2. Again, the two left columns pertain to more variable population configurations than the right columns. The likely explanation is that greater variance of population numbers translates ceteris paribus into a greater variety of the allocated voting weights. For a fixed quota, this tends give rise to more distinct winning coalitions and facilitates a better match between ideal probability vectors and those achievable by power laws. It is worth noting that the variances of $U(3 \cdot 10^6, 10^7)$ and the normal distribution $N(10^7, 2 \cdot 10^6)$ are

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<th>(II) $U(0, 10^8)$</th>
<th>(III) $U(3 \cdot 10^6, 10^7)$</th>
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Table 2: Optimal $\alpha$ for normally distributed constituency sizes

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<th>(II) $N(10^7, 4 \times 10^6)$</th>
<th>(III) $N(10^7, 2 \times 10^6)$</th>
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<td>(2.18 $\times 10^{-12}$)</td>
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</table>

Table 2: Optimal $\alpha$ for normally distributed constituency sizes

roughly the same, while a comparison of columns (III) and (IV) in Table 2 shows that the estimated probabilities are, in most cases, closer to their ideal values with uniformly than with normally distributed constituency sizes. So, even though variance of population sizes seems to have a beneficial effect on the achievable equality of representation for a given distribution type, statements involving distinct types require caution. Under a normal distribution, a majority of constituencies have similar sizes; the variance is associated mainly with outliers, which a uniform distribution does not produce. Minor population differences between non-outlier constituencies, however, cannot be easily balanced by weight-induced, discrete pivot probability differences. The precise value of $\alpha^*(q)$ and the quality of representation then may depend heavily on the constituency configuration and quota at hand, as in case of configuration (III) in Table 2.

Table 3 reports results for population sizes drawn from a Pareto distribution $P(\kappa, x)$. The parameter $\kappa > 0$ determines the skewness of the distribution, and $x > 0$ is the minimum possible value. Here, only a single or at most very few large constituencies exist, while most are small. The large ones are particularly disadvantaged by their often very central position when a supermajority rules applies. While high values of $\alpha$ would create an oligarchy or dictatorship in the corresponding weighted voting game and disenfranchise small members, moderate values give big outliers insufficiently great pivot probability. This explains rather low values of $\alpha$ under simple majority rule and comparatively high values

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20It is not possible to compare variances of constituency sizes because $P(\kappa, x)$ has infinite variance for $\kappa \leq 2$. 

18
Distribution of constituency sizes

<table>
<thead>
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Table 3: Optimal $\alpha$ for Pareto distributed constituency sizes

for the most demanding quotas.

### 5.2 EU Council of Ministers

The EU Council of Ministers decides the largest part of issues by qualified majority voting (the others by unanimity rule). A proposal is adopted according to the current rules, agreed in the Treaty of Nice, if its supporters wield majorities in three dimensions: first, 255 out of 345 votes ($\approx 73.9\%$) need to be cast in favor of the proposal; the number of votes allocated to each member state roughly reflects the square root of population size. Second, the supporters must represent a simple majority of the currently 27 EU member states. Third, any member state may ask for confirmation that the approving votes represent at least 62% of the EU’s total population. The latter two requirements are, however, almost negligible. They are in the great majority of cases fulfilled whenever the qualified majority is met (see Felsenthal and Machover 2001).

Figure 3 shows, the effect of a quota $q > 0.5$ on representation for EU27 population data.\(^{21}\) The respective best value of $\alpha \in \{0, 0.02, \ldots, 0.98, 1\}$ is represented by the solid graph and measured on the left vertical axis. The figure suggests that the optimal $\alpha$ is approximately a quadratic function of $q$. The right vertical axis measures the corresponding

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\(^{21}\) The focus on situations in which the status quo lies outside the Pareto subset of $X$ does not seem particularly narrow for the EU Council. For those issues that are prepared by the European Commission for a Council decision, it is very often not the direction of new legislation, but the extent of change which is subject to legislative bargaining and, eventually, voting.
cumulative individual quadratic deviation. Its origin is placed in the upper right corner, so that the corresponding dashed graph can be interpreted as closeness of fit between the (estimated) pivot probabilities under the optimal $\alpha$-rule and ideal egalitarian probabilities. The decline in closeness (or increase in deviation) confirms that representation becomes increasingly unequal as the quota is increased. For instance, the ideal probabilities in the Council are 0.08% and 16.62% for Malta and Germany, respectively. Already for $q = 0.7$, however, their pivot probabilities under the best elementary weight assignment rule $w_j = n_j^{0.62}$ are 0.06% and 15.45%, which is 25% and 7% short of the ideal.

6 Interim heterogeneity between constituencies

We have so far maintained the assumption that the ideal point of any representative $j \in \{1, \ldots, m\}$ coincides with the median of $n_j$ identically and independently distributed ideal points. The motivation for doing so was primarily normative: if constitutional rules are designed behind a “veil of ignorance”, any specific information on the likely issues and preferences which will prevail in front of the veil should be disregarded; asymmetries should only be acknowledged when they are a defining part of the allocation problem.

However, it might be acknowledged even in normative constitutional design that voters tend to be more similar within constituencies than between constituencies. This could be the result of a sorting process ("voting with one’s feet") à la Tiebout (1956). Or, as
Alesina and Spolaore (2003) argue, some preference homogeneity within a country develops over time due to geographical proximity and national policies fostering cultural uniformity. Generally, citizens of a given constituency will share historical experience, traditions, language, communication etc. – especially if constituencies correspond to entire countries, as in the EU case. This fact can be expected to induce a common set of values and common beliefs. The existence of diverse value systems across constituencies is plausibly the reason why they differ in population size and cannot easily be redistricted. Under the premise behind Penrose’s square root rule, that the preferences of all individual citizens are independently and identically distributed, there is little justification – apart from implementation costs perhaps – for not simply regrouping citizens into purely administrative districts with equal numbers of voters (or even a single one). This would altogether avoid the need for weighted voting at the top tier.

While correlations of voter ideal points from different countries or between groups within the same country appear to be rather volatile and should not be acknowledged in a priori analysis, the existence of country peculiarities as such seems a stable phenomenon. It can, therefore, be argued that the assumption that, for any fixed issue, voter preferences tend to have more in common within a constituency than across constituencies, is a defining part of the allocation problem at hand. It is relevant for the design of a fair constitution, and then need to be acknowledged behind the veil of ignorance. We will do so in this section. Before getting more specific, note that we still conduct a normative investigation of a priori power. It is to be distinguished from studies of a posteriori power, which, e.g., incorporate historical similarities or dissimilarities between EU countries based on economic or social dimensions (Widgrén 1995), size or geographical position (Beisbart and Hartmann 2006), or opinion surveys (Passarelli and Barr 2007).

We now introduce a particular form of dependence within constituencies, which maintains the assumption of a priori identically distributed ideal points for all citizens. Specifically, consider the following kind of interim heterogeneity between constituencies: Let any given policy issue be associated with a constituency-specific shock $\mu_j$ with $E[\mu_j] = 0$, which is distributed according to the same distribution $H$ with continuous density function $h$ for all constituencies. Conditional on $\mu_j = 0$, let constituency $C_j$’s citizens have i.i.d. ideal points with density $f$, exactly as in the baseline model. Under this condition, also the policy position of $C_j$’s representative, $\lambda_j$, corresponds exactly to the baseline case studied in Section 3 – it is still the median of $n_j$ i.i.d. ideal points with density $f$.

In general, however, i.e., conditional on a realization of $\mu_j$ that possibly differs from zero, let the individuals in $C_j$ have i.i.d. ideal points which come from distribution $F$ shifted by $\mu_j$. Namely, for any $j = 1, \ldots, m$, let the ideal point $\nu^j_i$ of any individual $i \in C_j$ be the sum of the (interim) realized constituency-specific shock $\mu_j$ and a draw from $f$. The median and mean of the interim distribution of the ideal points within $C_j$, therefore, equals $\mu_j$. Ideal points $\nu^j_i$ and $\nu^j_l$ are interim (after the shock realization in the constituency) closer to each other than to the corresponding ideal points in constituency $C_k$, where a different

---

22A “common belief” is also motivating Straffin’s (1977) homogeneity assumption, for which the probability of a voter affecting a binary collective decision coincides with the Shapley-Shubik index.
realization $\mu_k$ from distribution $H$ reflects a different general opinion on the issue at stake. Ideal points are still identically distributed across constituencies ex ante; but dependence gives rise to different interim distributions.

The density $f_{\lambda_j}$ of the respective median position $\lambda_j$ in $C_j$ is the convolution of two density functions. First, it involves a baseline density $f^0_{\lambda_j}$, which coincides with $f_{\lambda_j}$ in the previous sections, and for $\mu_j = 0$ given. Second, it involves $h$:

$$f_{\lambda_j}(x) = \int_{-\infty}^{\infty} f^0_{\lambda_j}(x - \mu_j) h(\mu_j) d\mu_j.$$  \hspace{1cm} (8)

That the distributions of $\lambda_j$ differed near their common mean (and median) for $j = 1, \ldots, m$, and more specifically that $f_{\lambda_j}(0)$ varied linearly in $\sqrt{\mu_j}$, was the key property leading to a square root rule in Proposition 2. Some such differences must, of course, be maintained also in case of dependent ideal points, because different population sizes give rise to different baseline densities $f^0_{\lambda_j}$. However, these may be of second order in comparison. In particular, if the standard deviation $\sigma^0_{\lambda_j}$ of distribution $F^0_{\lambda_j}$ (i.e., the c.d.f. of $\lambda_j$ conditional on $\mu_j = 0$) is sufficiently small compared to the degree of interim heterogeneity, as captured by the standard deviation $\sigma_H$ of $H$, then $\lambda^0_j$ can almost be regarded as a constant, and $\lambda_j = \lambda^0_j + \mu_j$ is determined essentially by distribution $H$.

For illustration, let $H$ be a uniform distribution on $[-a, a]$ for $a > 0$. Then (8) becomes

$$f_{\lambda_j}(x) = \frac{1}{2a} \int_{-a}^{a} f^0_{\lambda_j}(x - \mu_j) d\mu_j = \frac{1}{2a} \left[ F^0_{\lambda_j}(x + a) - F^0_{\lambda_j}(x - a) \right].$$

The standard deviation of $F^0_{\lambda_j}$ is small if the considered constituency $C_j$ has a large population. Therefore, $F^0_{\lambda_j}(x + a) \approx 1$ and $F^0_{\lambda_j}(x - a) \approx 0$ if $a \gg 0$ and $x$ is small. It follows that $f_{\lambda_j}(x) \approx 1/(2a)$ in the center of the interval $[-a, a]$. There it does not depend on the specific constituency $C_j$ which is considered (assuming $n_j$ is large for all $j \in \{1, \ldots, m\}$). Only for $x$ close to the boundaries of $\mu_j$’s support will $f_{\lambda_j}(x)$ and $f_{\lambda_k}(x)$ differ. This is illustrated by Figure 4. Densities $f^0_{\lambda_A}$ and $f^0_{\lambda_B}$ are normal with standard deviation $\sigma_A = 0.08$ (blue graph) and $\sigma_B = 0.12$ (red dashed graph), respectively, and heterogeneity is induced by a uniform distribution of $\mu_j$ on $[-1, 1]$.

A very similar picture would result if, for instance, $H$ were a normal distribution with zero mean and variance $\sigma_H^2$. Then $\lambda_j$ is normally distributed with zero mean and variance $\sigma^0_j + \sigma_H^2$. Provided that $\sigma_H \gg \sigma_{\max} \equiv \max\{\sigma^0_1, \ldots, \sigma^0_m\}$ we have

$$\lambda_j \sim N(0, \sqrt{\sigma^2_j + \sigma^2_H}) \approx N(0, \sqrt{\sigma^2_k + \sigma^2_H}) \sim \lambda_k$$

for any $j, k \in \{1, \ldots, m\}$. This and Proposition 1 imply

**Corollary 2.** Under the conditions of Proposition 1 and assuming that the ideal point of any representative $j \in \{1, \ldots, m\}$ coincides with the median of $n_j$ ideal points $\nu^0_j = \mu_j + \kappa_i$
Figure 4: Densities $f_{\lambda_A}(x)$ and $f_{\lambda_B}(x)$ for constituency-specific shocks $\mu_j \sim U(-1,1)$

where $\mu_j$ is drawn independently from $H$, $\kappa_i$ is drawn independently from $F$ for every $i \in C_j$, and $\text{Var}[\mu_j] \gg \text{Var}[\kappa_i]$,

$$\pi_i(R^{(m)}) \approx \frac{w_i}{\sum_{k=1}^{m} w_k}$$

for large $m$.

We, therefore, have

**Proposition 3.** Under the conditions of Proposition 1 and assuming that the ideal point of any representative $j \in \{1, \ldots, m\}$ coincides with the median of $n_j$ ideal points $\nu_j \equiv \mu_j + \kappa_i$ where $\mu_j$ is drawn independently from $H$, $\kappa_i$ is drawn independently from $F$ for every $i \in C_j$, and $\text{Var}[\mu_j] \gg \text{Var}[\kappa_i]$, representation is approximately egalitarian for large $m$ if weights $w_1, \ldots, w_m$ are chosen proportional to $n_1, \ldots, n_m$.

Proposition 1 contains a statement about $\pi_j(R^{(m)})$, i.e., the probability of representative $j$ being the pivotal representative in $R^{(m)}$ under *simple majority rule*, i.e., a relative quota of $q = 0.5$; this corresponds to $\lambda_j$ being the weighted median of ideal points $\lambda_1, \ldots, \lambda_m$. We saw in Section 5 that extending the corresponding limit result to supermajority rules was difficult. In particular, the square root rule of Proposition 2 fails to extend to relative voting thresholds $q > 0.5$ in the i.i.d. case. The situation is different if interim heterogeneity between constituencies is considered. Namely, for $\sigma_H \gg \sigma_{\text{max}}$, the distributions $F_{\lambda_j}$ and $F_{\lambda_k}$ are essentially equal for an entire neighborhood of their common mean and median. This means that, for all practical purposes, the orderings of $\lambda_1, \ldots, \lambda_m$ are equiprobable in the range in which a majority threshold $0.5 < q < 1$ is typically passed. In other words, denoting the pivot probability of representative $j$ for decision quota $0.5 < q < 1$ by $\pi_j(q; R^{(m)})$, we have

$$\pi_j(q; R^{(m)}) \approx SSI_j(q; w_1, \ldots, w_m).$$

(9)
How close $q$ can be to the unanimity threshold of 100% depends on how dominant the between-constituency heterogeneity is relative to the within-constituency type. Approximation (9) becomes inaccurate for $x$ far from 0.

This reasoning is made more precise in Appendix C for uniform shock distributions. We can conclude three things. First, replacing the clause “weights $w_1, \ldots, w_m$ are chosen proportional to $n_1, \ldots, n_m$” in Proposition 3 by “weights $w_1, \ldots, w_m$ such that the resulting Shapley-Shubik index is (approximately) proportional to $n_1, \ldots, n_m$” will relax the qualification “for large $m$” because of (9). Second, we do not need to appeal to the conditions of Proposition 1 because the SSI automatically takes care of any combinatorial particularities. Extremely skewed or similar constituency sizes may mean that the inverse problem (of finding weights which induce the desired SSI vector) has no particularly good solution. But the obtained allocation is still as egalitarian as possible. Third, and most importantly, the mentioned linear SSI-based voting weight allocation rule also applies if supermajority rule is used in $\mathcal{R}^{(m)}$.

The latter is particularly important for applications to the EU Council. Figure 5 depicts those SSI values of the 27 EU member states which would arise if the best unconstrained weights $w^*$ (see Section 5) for different values of $\sigma_H$ were used. A linear Shapley-Shubik rule would have directly delivered the depicted, approximately egalitarian representation – without need to search for the optimal exponent $\alpha$. Still, a small caveat is needed: implementing a linear SSI-based allocation rule perfectly would require an exact solution to
the inverse problem. This, however, need not exist – especially, if a supermajority quota
has already been fixed (cf. Figure 1 and, moreover, recall the generic impossibility of egal-
itarian representation for unanimity rule) at the top tier. This problem can be detected in
Figure 5: for a 50%-quota, i.e., simple majority rule, the Shapley-Shubik indices associated
with best unconstrained weights are located nicely on the 45°-line. But for a 73.9%-quota,
which reflects the Treaty of Nice’s provision, they meander around that line.

Figure 6 shows the optimal exponent $\alpha$ for elementary power law allocation rules as a
function of $\sigma_H/\sigma_{max}$. It indicates a rapid transition from a square root rule being most
egalitarian to a near-linear rule being optimal. The square root rule breaks down already
for small degrees of heterogeneity.

7 Concluding remarks

This paper has addressed the issue of egalitarian representation of individuals in a two-tier
voting system, such as the EU Council or the US Electoral College. Our concern was the
equalization of the indirect influence that bottom-tier voters can be expected to have on
the collective decision in case of a one-dimensional convex policy space. The square root
rule has played a prominent role in the political discussion in the EU as well as the scientific
discussion of binary policy environments. It was suggested to apply also more generally by
Maaser and Napel (2007), and now has a sound analytical foundation in the median voter
world. However, the rule turned out to be considerably less robust than initially thought:
it does not extend to supermajority rules; it does badly in case of positive correlation at
the constituency level.
The latter should, in some sense, not be very surprising. The extensive literature on optimal voting weight allocations for dichotomous policy alternatives has, for various objective functions, brought about either a square root or a linear rule (with few exceptions). Square root rules typically follow from far-reaching homogeneity and independence assumptions, but a linear rule is called for in case of dependence and between-constituency heterogeneity. For instance, Kirsch (2007) finds square root weights to minimize the extent of disagreement between the council’s binary decision and the popular vote with independent voters, but a linear rule if a sufficiently strong “collective bias” of the voters within each constituency is introduced. The utilitarian approach of Barberà and Jackson (2006) calls for square root weights in the mentioned “fixed-size-of-blocks model”, while they derive a linear rule in a “fixed-number-of-blocks model”, which divides each constituency into the same number of blocks of identical voters. Beisbart and Bovens (2007) come to a very similar conclusion when trying to maximize citizens’ expected utility in their model: with i.i.d. utility parameters and simple majority rule, square root weights maximize total expected utility (and equalize it across citizens). But if an individual’s utility is correlated with more other individuals the larger his constituency is, then the square root rule quickly makes way for a proportional one.23

So, in a very different framework, this paper echoes a general finding that has emerged from the literature on two-tier voting systems for binary alternatives: ex ante independent and identical voters call for a voting weight allocation rule based on the square root of population sizes, as originally argued by Penrose (1946). However, sufficiently strong affiliation within constituencies renders most people’s basic intuition correct – plain proportionality does the trick.

Appendix A

This appendix reports a heuristic investigation of top-tier voting weights which are egalitarian in a welfarist sense, i.e., equalize utility from rather than influence on policy outcomes. For this, we have to suppose that all voters have cardinal utility which is interpersonally comparable. Specifically, let all individuals have preferences over policy outcomes \( x \in X \) of the von Neumann-Morgenstern type, and let these be representable by a quadratic form (common in spatial voting models):

\[
    u_i(x, \nu^j) = -(x - \nu^j)^2
\]

for all \( i \in C_j \) and \( j = 1, \ldots, m \). For any given voter \( i \), let \( \omega_i \equiv \Pr(j = P : m \land \lambda_j = \nu_j) = \pi_j(R^{(m)})/n_j \) denote the probability of being doubly pivotal, i.e., of \( i \) being the median in \( C_j \) and representative \( j \) being the weighted median in \( R^{(m)} \). The expected utility of \( i \) can

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23 It needs to be replaced by an equal weight allocation if the objective is to equalize all individuals’ expected utilities.
then be decomposed as follows

\[
\mathbb{E}u_i = \varpi_i \cdot \mathbb{E}(u_i \mid i \text{ is doubly pivotal}) + \varpi_l \cdot \mathbb{E}(u_i \mid l \text{ is doubly pivotal}, l \in C_k, k \neq j) + \sum_{h \in N, h \neq i, l} \varpi_h \cdot \mathbb{E}(u_i \mid h \text{ is doubly pivotal}). \tag{11}
\]

Now consider the case in which all ideal points are i.i.d. and come from a standard normal distribution, i.e., let \( F = \Phi \) and \( f = \phi \). The median \( \lambda_j \) in \( C_j \) is approximately normally distributed with mean 0 and variance

\[
\sigma^2_{\lambda_j} = \frac{1}{2 \phi(\Phi^{-1}(0.5)) \sqrt{\pi}} = \frac{\pi}{2n_j^2}, \tag{12}
\]

(note that \( \phi(\Phi^{-1}(0.5)) = 1/(\sqrt{2\pi}) \)). The policy outcome \( x \) is assumed to always coincide with the median ideal point of some constituency \( k \in \{1, \ldots, m\} \). In a heuristic approximation which ignores the combinatorial features of discrete weight distributions by directly considering the expected value of the aggregate weight of representatives \( l \neq k \) with ideal points \((−\infty, x)\), the policy outcome conditional on representative \( k \) being pivotal is a random variable which has a doubly truncated normal distribution with mean zero, variance given by (12) and truncation points

\[
z_k \approx \frac{w_k}{2 \sum_{j \neq k} w_j f_{\lambda_j}(0)}
\]

and \(-z_k\). Denote this policy outcome by \( x_k \). The density of \( x_k \)'s distribution is

\[
g(x_k \mid -z_k, z_k) = \frac{1}{\int_{-z_k}^{z_k} \frac{1}{\sigma_{\lambda_k} \sqrt{2\pi}} \exp \left( -\frac{y^2}{2\sigma^2_{\lambda_k}} \right) dy} \frac{1}{\sigma_{\lambda_k} \sqrt{2\pi}} \exp \left( -\frac{x_k^2}{2\sigma^2_{\lambda_k}} \right)
\]

Using (11), one can now compare the expected utilities \( \mathbb{E}u_i \) and \( \mathbb{E}u_l \) of voters \( i \in C_j \) and \( l \in C_k \). Consider the case that the outcome \( x \) is determined by \( h \in \{1, \ldots, m\}, h \neq i, l \). Then, given the i.i.d. assumption, the expected distance between \( \nu_i^l \) and \( x \) is the same as between \( \nu_k^l \) and \( x \). Thus, for any pair \( i, l \) of individuals we have

\[
\sum_{h \in N, h \neq i, l} \varpi_h \cdot \mathbb{E}(u_i \mid h \text{ is doubly pivotal}) = \sum_{h \in N, h \neq i, l} \varpi_h \cdot \mathbb{E}(u_l \mid h \text{ is doubly pivotal}).
\]

Thus, we obtain \( \mathbb{E}u_i = \mathbb{E}u_l \) if and only if

\[
\varpi_l \cdot \mathbb{E}(u_i \mid l \text{ is doubly pivotal}, l \in C_k, k \neq j) = \varpi_i \cdot \mathbb{E}(u_l \mid i \text{ is doubly pivotal}, i \in C_j, j \neq k). \tag{13}
\]

Intuitively, the expected utility of voter \( i \in C_j \) when \( l \in C_k \) is doubly pivotal is subject to two influences: First, different population sizes of constituencies \( j \) and \( k \) imply that
the respective median voters scatter more or less around the mean of zero. The expected distance of any given individual ideal point \( \nu_i^j \) to \( \lambda_k \) is the greater, the smaller constituency \( k \) (see (12)). This calls for giving more weight to a constituency the larger its population size. Second, the relative voting weight of constituency \( C_k \) enters \( E \) through the truncation points \( -z_k \) and \( z_k \). The larger \( k \)'s weight, the wider the interval \([−z_k, z_k]\), which increases \( \pi_i(\mathcal{R}^{(m)}) \) (and thus \( \varpi_i \)) as well as the expected distance between the individual ideal point \( \nu_i^j \) and \( x_k \). This latter effect implies that it cannot be optimal to give all weight to the most populous constituency. In order to satisfy the utility-egalitarian condition (13) the voting weights have to balance these forces.

Building on the analysis in Section 3, one can derive a rule which approximately equalizes expected utility across individual voters. The expected utility of voter \( i \) with ideal point \( \nu_i^j \) in case that a member of constituency \( C_k \) is doubly pivotal is given by

\[
E u_i(x_k, \nu_i^j) = -\int_{-\pi}^\pi \int_{-\pi}^\pi (x_k - \nu_i^j)^2 f(\nu_i^j) g(x_k | -z_k, z_k) d\nu_i^j dx_k.
\]

After some tedious calculations, one can obtain that

\[
E u_i(x_k, \nu_i^j) = -1 - \sigma_{\lambda_k}^2 + \frac{2z_k \sigma_{\lambda_k} \phi\left(\frac{z_k}{\sigma_{\lambda_k}}\right)}{\Phi\left(\frac{z_k}{\sigma_{\lambda_k}}\right) - \Phi\left(\frac{-z_k}{\sigma_{\lambda_k}}\right)}.
\]

Observing that

\[
\varpi_i = \Pr(k = P; m)/n_k \approx \frac{1}{n_k \sigma_{\lambda_k}} \left[ \Phi\left(\frac{z_k}{\sigma_{\lambda_k}}\right) - \Phi\left(\frac{-z_k}{\sigma_{\lambda_k}}\right) \right],
\]

\( \varpi_i E u_i(x_k, \nu_i^j) \) can be written as

\[
\frac{1}{\sqrt{n_k}} \sum_{\ell \neq k} w_k \sqrt{n_{\ell}} w_{\ell} - \frac{1}{n_k} \sum_{\ell \neq k} w_k \sqrt{n_{\ell}} w_{\ell} \sqrt{\frac{\pi}{2}} \exp\left(-\frac{\pi n_k w_k^2}{2 \sum_{\ell \neq k} \sqrt{n_{\ell}} w_{\ell}}^2\right) + \frac{1}{(\sqrt{n_k})^3} \sum_{\ell \neq k} w_k \sqrt{n_{\ell}} w_{\ell} \left(\sqrt{\frac{\pi}{2}}\right)^2.
\]

Finally, focusing on weight allocation rules of the power-law type, i.e., \( w_k = n_k^\alpha \), the utility-egalitarian condition (13) becomes

\[
\frac{n_k^{\alpha-0.5}}{\sum_{\ell \neq k} n_{\ell}^{\alpha+0.5}} \left[ 1 - \sqrt{\frac{\pi}{2n_k}} \exp\left(-\frac{-\pi n_k^{2\alpha+1}}{2 \sum_{\ell \neq k} n_{\ell}^{\alpha+0.5}}^2\right) + \frac{\pi}{2n_k} \right]
\]

\[
= \frac{n_j^{\alpha-0.5}}{\sum_{\ell \neq j} n_{\ell}^{\alpha+0.5}} \left[ 1 - \sqrt{\frac{\pi}{2n_j}} \exp\left(-\frac{-\pi n_j^{2\alpha+1}}{2 \sum_{\ell \neq j} n_{\ell}^{\alpha+0.5}}^2\right) + \frac{\pi}{2n_j} \right].
\]

Equation (14) defines the approximately optimal \( \alpha \) for any pair of constituencies \( k, j \in \{1, \ldots, m\} \). Provided that each constituency \( C_k \) is ‘small’ compared to the sum of populations other than \( C_k \)'s, the exponential function in (14) assumes a value slightly less
than 1. In order to make the expected utility of citizens of constituencies of different sizes approximately equal, the dominating influence of terms $n_k^{\alpha-0.5}/\sum_{\ell \neq k} n_{\ell}^{\alpha+0.5}$ and $n_j^{\alpha-0.5}/\sum_{\ell \neq j} n_{\ell}^{\alpha+0.5}$ must be eliminated. This is achieved by choosing $\alpha = 0.5$.

Appendix B

The premise in Proposition 1 considers an arbitrary but fixed number $r$ of “prototypes” for the infinite sequence of representatives in councils $R^{(1)} \subset R^{(2)} \subset R^{(3)} \subset \ldots$. Namely, there is some mapping $\tau: \mathbb{N} \to \{1, \ldots, r\}$ such that $w_j = \tilde{w}_{\tau(j)}$ and $f_{\lambda_j} = \tilde{f}_{\tau(j)}$. The respective density at the common mean and median of zero is bounded by $u = \min_{s \in \{1, \ldots, r\}} \tilde{f}_s(0) > 0$ and $\bar{u} = \max_{s \in \{1, \ldots, r\}} \tilde{f}_s(0) \leq u$. The proportion of representatives of a given type $s \in \{1, \ldots, r\}$ in $R^{(m)}$ approaches a fraction $\beta_s > 0$. Denote the minimal and maximal such fractions by $\beta$ and $\bar{\beta}$.

Let $\pi^m_s$ denote the probability that the weighted median of the sequence $\lambda_1, \ldots, \lambda_m$ is of type $s$. (W.l.o.g. we can assume that a unique median is taken and that all realizations of the $\lambda_j$ are different.) By $\pi^m_s(\varepsilon)$ we denote the probability that the weighted median of the sequence $\lambda_1, \ldots, \lambda_m$ is of type $s$ and falls inside the interval $[-\varepsilon, \varepsilon]$. By $\hat{\pi}^m_s(\varepsilon)$ we denote the probability that the weighted median is of type $s$ conditional on falling inside interval $[-\varepsilon, \varepsilon]$. (So $\sum_s \hat{\pi}^m_s(\varepsilon) = 1$, while typically $\sum_s \pi^m_s(\varepsilon) < 1$.)

We will consider a particular choice of $\varepsilon$, namely $\varepsilon(m) \equiv m^{-\frac{1}{3}}$, in the following. Our first aim is to prove that

$$\lim_{m \to \infty} \pi^m_s = \lim_{m \to \infty} \pi^m_s(\varepsilon(m)) = \lim_{m \to \infty} \hat{\pi}^m_s(\varepsilon(m)) > 0$$

for all $s \in \{1, \ldots, r\}$.

Choose $\varepsilon_2 > 0$ such that

$$\frac{1}{2} u \leq \tilde{f}_s(x) \leq 2 \bar{u}$$

for all $s \in \{1, \ldots, r\}$ and all $x \in [-\varepsilon_2, \varepsilon_2]$, and choose $\varepsilon_3 > 0$ such that

$$\frac{1}{2} \beta \leq \beta_s(m) \leq 2 \bar{\beta}$$

for all $s \in \{1, \ldots, r\}$ and all $m \geq \left\lceil \frac{1}{\varepsilon_3^3} \right\rceil$. The probability of an ideal point of type $s$ to fall inside $[-\varepsilon, \varepsilon]$ is

$$p_s(m) \equiv \int_{-\varepsilon(m)}^{\varepsilon(m)} \tilde{f}_s(x) dx$$

(18)
and satisfies
\[ um^{-\frac{1}{3}} = u\varepsilon(m) \leq p_s(m) \leq 4\pi\varepsilon(m) = 4\pi m^{-\frac{1}{3}}. \] (19)

In the following, we will appeal to Hoeffding’s inequality\(^{24}\) in order to obtain bounds on the probability that the average \( \overline{X} \equiv \frac{1}{n} \sum_{i=1}^{n} X_i \) of \( n \) independent bounded random variables \( X_i \in [a_i, b_i] \) is located more than some distance \( t > 0 \) away from its expectation. Namely, the inequality guarantees
\[
\Pr \{ |\overline{X} - E[\overline{X}]| > t \} \leq 2 \exp \left( -\frac{2t^2n^2}{\sum_{i=1}^{n} (b_i - a_i)^2} \right). \] (20)

We will invoke the inequality for specific sequences of random variables \( X_1, \ldots, X_m \) (namely, shares of ideal points with particular properties), where each \( X_i \in [0, 1] \). Inequality (20) in this case simplifies to
\[
\Pr \{ |\overline{X} - E[\overline{X}]| > t \} \leq 2 \exp \left( -2t^2m \right). \] (21)

As \( t \), we will choose \( t = c \cdot m^{-\frac{5}{6}} \) for a suitably small constant \( c \), so that
\[
\Pr \{ |\overline{X} - E[\overline{X}]| > cm^{-\frac{5}{6}} \} \leq 2 \exp \left( -2c^2m^\frac{5}{6} \right) \equiv P_1(m). \] (22)

\( P_1(m) \) exponentially goes to zero.

Now let us focus on those \( \lambda_i \) with \( \tau(i) = s \) for a given \( s \in \{1, \ldots, r\} \). For given \( m \), let \( m_s \leq m \) denote their number in \( \lambda_1, \ldots, \lambda_m \). Since we have \( \beta_s(m) \geq \frac{1}{2} \beta > 0 \) for \( m \geq \left\lceil \frac{1}{\varepsilon^3} \right\rceil \), it must be the case that \( m_s \geq \frac{1}{2} \beta m \). From Hoeffding’s inequality we can conclude:

(I) Denoting the number of representatives of type \( s \) with an ideal point realization \( \lambda_j \in (-\infty, -\varepsilon(m)) \) “on the left” by \( m_s^L \), we have
\[
\Pr \left\{ \frac{m_s^L}{m} \in \left[ \frac{1-p_s(m)}{2} \cdot \beta_s(m) - cm^{-\frac{5}{6}}, \frac{1-p_s(m)}{2} \cdot \beta_s(m) + cm^{-\frac{5}{6}} \right] \right\} \geq 1 - P_1(m_s).
\]

(II) Denoting the number of representatives of type \( s \) with an ideal point realization \( \lambda_j \in [-\varepsilon(m), \varepsilon(m)] \) “in the middle” by \( m_s^M \), we have
\[
\Pr \left\{ \frac{m_s^M}{m} \in \left[ p_s(m) \cdot \beta_s(m) - cm^{-\frac{5}{6}}, p_s(m) \cdot \beta_s(m) + cm^{-\frac{5}{6}} \right] \right\} \geq 1 - P_1(m_s).
\]

\(^{24}\)See Hoeffding (1963, Theorem 2).
(III) Denoting the number of representatives of type s with an ideal point realization \( \lambda_j \in (\varepsilon(m), \infty) \) “on the right” by \( m_s^R \), we have

\[
\Pr \left\{ \frac{m_s^R}{m} \in \left[ \frac{1 - p_s(m)}{2} \cdot \beta_s - cm^{-\frac{2}{s}}, \frac{1 - p_s(m)}{2} \cdot \beta_s + cm^{-\frac{2}{s}} \right] \right\} \geq 1 - P_1(m_s).
\]

Let \( \varepsilon' = \min \{ \varepsilon_1, \varepsilon_2, \varepsilon_3 \} \) and consider \( m \geq \left\lceil \frac{1}{2\varepsilon'} \right\rceil \) in the following.

**Lemma 1.** For \( c < \frac{1}{10}u\beta \) we have \( m_s^L < \frac{1}{2}\beta_s m \) and \( m_s^R < \frac{1}{2}\beta_s m \) with a respective probability of at least \( 1 - P_1(m_s) \).

**Proof.** With probability at least \( 1 - P_1(m_s) \), we have

\[
m_s^L \leq m \cdot \left( \frac{1 - p_s(m)}{2} \cdot \beta_s + cm^{-\frac{2}{s}} \right) \leq \frac{1}{2}\beta_s m - \frac{1}{2}\beta_m \cdot m^2 + cm^2.
\]

Inserting the upper bound for \( c \) yields the stated result for \( n_s^L \). The result for \( n_s^R \) follows analogously.

**Lemma 2.** For \( c < \frac{1}{10}u\beta \) we have \( m_s^L + m_s^M > \frac{1}{2}\beta_s(m)m + \frac{1}{3}m_s^M \) and \( m_s^R + m_s^M > \frac{1}{2}\beta_s(m)m + \frac{1}{3}m_s^M \) with a respective probability of at least \( 1 - P_1(m_s) \).

**Proof.** With probability at least \( 1 - P_1(m_s) \), we have

\[
m_s^L + \frac{2}{3}m_s^M \geq m \left( \frac{1 - p_s(m)}{2} \cdot \beta_s(m) - cm^{-\frac{2}{s}} \right) + \frac{2}{3}m \left( p_s(m) \cdot \beta_s(m) - cm^{-\frac{2}{s}} \right)
\]

\[
= \frac{1}{2}\beta_s(m)m + \frac{1}{6}p_s(m)\beta_s(m)m - \frac{5}{3}cm^2
\]

\[
\geq \frac{1}{2}\beta_s(m)m + \frac{1}{6}u\beta_m m^2 - \frac{5}{3}cm^2,
\]

from which we conclude the stated inequality by inserting the upper bound for \( c \). A similar calculation yields the second stated result.

**Corollary 3.** \( \lim_{m \to \infty} \pi_s^m = \lim_{m \to \infty} \hat{\pi}_s^m(\varepsilon(m)) = \lim_{m \to \infty} \hat{\pi}_s^m(\varepsilon(m)) > 0 \) for all \( s \in \{1, \ldots, r\} \).

**Proof.** With probability at least \( [1 - P_1(\text{\text{min}}\{m_1, \ldots, m_s\})]^{3r} \), the conditions considered in (I), (II), and (III) are simultaneously fulfilled for all \( s \in \{1, \ldots, r\} \). By the previous two lemmas, the median of all ideal points of type \( s \) is taken inside the interval \( [-\varepsilon(m), \varepsilon(m)] \) in such a situation for every \( s \in \{1, \ldots, r\} \). Therefore, also the weighted median of \( \lambda_1, \ldots, \lambda_m \) must fall inside this interval with probability at least \( [1 - P_1(\text{\text{min}}\{m_1, \ldots, m_s\})]^{3r} \). This probability converges to one as \( m \), and hence \( \text{\text{min}}\{m_1, \ldots, m_s\} \), go to infinity.

Now suppose that \( \hat{f}_s(x) = g_j \) for all \( s \in \{1, \ldots, r\} \) and all \( x \in [-\varepsilon(m), \varepsilon(m)] \), where \( m \) is suitably large. Again we consider the situation in which the conditions considered in (I), (II), and (III) are simultaneously fulfilled for all \( s \in \{1, \ldots, r\} \). Due to Lemmas 1 and
2, the weighted median of \( \lambda_1, \ldots, \lambda_m \) falls into the interval \([-\varepsilon(m), \varepsilon(m)]\). It corresponds to the pivotal representative amongst those representatives \( j \) with \( \lambda_j \in [-\varepsilon(m), \varepsilon(m)] \) for a suitably chosen quota (it must lie between \( \frac{1}{3} \) and \( \frac{2}{3} \) in view of Lemma 2). For any given realization of the ideal points outside \([-\varepsilon(m), \varepsilon(m)]\), the corresponding quota can be computed. Then taking any such quota as given, we can focus on the restricted weighted voting game amongst only the remaining representatives. We have a bound on the number \( b_s(m) \) of involved representatives with \( \lambda_j \in [-\varepsilon(m), \varepsilon(m)] \) and type \( s \in \{1, \ldots, r\} \), namely

\[
|b_s(m) - \beta_s \cdot 2g_s \cdot m^3| \leq c \cdot m^3
\]  

(23)

is satisfied with a probability approaching one as \( m \to \infty \). All orderings of the participating players are equiprobable, because the conditional densities equal \( 1/2\varepsilon(m) \) on \([-\varepsilon(m), \varepsilon(m)]\) for all types \( s \in \{1, \ldots, r\} \). The respective pivot probability of a given representative conditioned to be inside \([-\varepsilon(m), \varepsilon(m)]\) therefore corresponds to his Shapley-Shubik index (SSI) in the restricted weighted voting game. As Lindner and Machover (2004, Theorem 2.3) have shown, based on Neyman (1982, Theorem 9.8), the SSI value of any representative \( j \) becomes proportional to \( w_j \) as the number of considered players (drawn from a finite number \( r \) of “prototypes”) approaches infinity, irrespective of the considered quota. The aggregate probability of some representative of type \( s \) to be pivotal in the restricted game becomes proportional to \( b_s(m)\tilde{w}_s \). It follows that we must have

\[
\lim_{m \to \infty} \pi^m_s(\varepsilon(m)) = \frac{\beta_s g_s \tilde{w}_s}{\sum_{t=1}^{r} \beta_t \tilde{g}_t \tilde{w}_t}.
\]

The next step is to relax the requirement that \( \tilde{f}_s(x) \) equals some constant \( g_s \) on \([-\varepsilon(m), \varepsilon(m)]\). For a given \( m \) we consider (tight) bounds \( g_s \leq \tilde{f}_s(x) \leq \overline{g}_s \) for all \( s \in \{1, \ldots, r\} \) and all \( x \in [-\varepsilon(m), \varepsilon(m)]\). Using these lower and upper bounds we have

\[
\lim_{m \to \infty} \frac{\beta_sg_s \tilde{w}_s}{\sum_{t=1}^{r} \beta_t \overline{g}_t \tilde{w}_t} \leq \lim_{m \to \infty} \pi^m_s(\varepsilon(m)) \leq \lim_{m \to \infty} \frac{\beta_sg_s \tilde{w}_s}{\sum_{t=1}^{r} \beta_t \overline{g}_t \tilde{w}_t}
\]  

(24)

As \( \varepsilon(m) \) vanishes, both \( g_s \) and \( \overline{g}_s \) tend to \( \tilde{f}_s(0) \). Thus we have

\[
\lim_{m \to \infty} \pi^m_s = \lim_{m \to \infty} \pi^m_s(\varepsilon(m)) = \frac{\beta_s \tilde{f}_s(0) \tilde{w}_s}{\sum_{t=1}^{r} \beta_t \tilde{f}_t(0) \tilde{w}_t}.
\]  

(25)

Finally, we can move from the probability of some representative of \( s \) being pivotal in \( R^{(m)} \), namely \( \pi^m_s \), to the probability of a specific representative \( j \) with \( \tau(j) = s \) being pivotal. Because all representatives of type \( s \) are symmetric,

\[
\pi_j(R^{(m)}) = \frac{\pi^m_{R(j)}}{m_{R(j)}}.
\]
It follows that
\[
\lim_{m \to \infty} \frac{\pi_j(R^{(m)})}{\pi_j(R^{(m)})} = \lim_{m \to \infty} \frac{\pi_j^{(m)}}{\pi_j^{(m)}} \cdot m_{\tau(j)} = \lim_{m \to \infty} \frac{\beta_{\tau(i)} \tilde{f}_{\tau(i)}(0) \tilde{w}_{\tau(i)}}{\beta_{\tau(j)} f_{\tau(j)}(0) \tilde{w}_{\tau(j)}} \cdot \frac{\beta_{\tau(j)}}{\beta_{\tau(i)}} = \frac{w_i f_{\lambda_j}(0)}{w_j f_{\lambda_j}(0)}
\]
as claimed in Proposition 1.

Appendix C

We can make the intuition developed in Section 6 for the case of sufficiently big interim heterogeneity between constituencies (relative to the common ex ante heterogeneity within constituencies, which interacts with population sizes to determine \(\sigma_{\text{max}}\)) more precise, if \(h\) is the density of a uniform distribution:\(^{25}\)

**Lemma 3.** Assume that the median \(\mu_j\) of \(F_{\lambda_j}\) is uniformly distributed on the interval \([-a, a]\) for \(j \in \{1, \ldots, m\}\). Fix \(k > 0\) such that \(a > k \sigma_{\text{max}} \equiv \bar{b}\), and then let \(b_j = k \sigma_j\).

Using
\[
f_{\lambda_j}(x) = \begin{cases} 
\frac{1}{a} [F_{\lambda_j}^0(b_j) - F_{\lambda_j}^0(-b_j)] & \text{if } -a + b_j \leq x \leq a - b_j \\
0 & \text{if } x < -a + b_j \text{ or } x > a - b_j 
\end{cases}
\]

(\(j = 1, \ldots, m\)) as approximations for the true densities \(f_{\lambda_j}\), the constituency-specific absolute approximation error \(\epsilon_j\) is given by
\[
\epsilon_j = 1 - \left(1 - \frac{b_j}{a}\right) \left[F_{\lambda_j}^0(b_j) - F_{\lambda_j}^0(-b_j)\right] \leq \epsilon_{\text{max}} = 1 - \left(1 - \frac{\bar{b}}{a}\right) \left[F_{\lambda_j}^0(\bar{b}) - F_{\lambda_j}^0(-\bar{b})\right] \quad (26)
\]

if \(C_a\) is the constituency with smallest population size.

**Proof.** First note that (8) can be decomposed as follows:
\[
f_{\lambda_j}(x) = \int_{-\infty}^{\infty} f_{\lambda_j}^0(x - \mu) h(\mu) d\mu = \int_{-\infty}^{\infty} f_{\lambda_j}^0(\mu) h(x - \mu) d\mu
\]
\[
= \int_{-\infty}^{\infty} f_{\lambda_j}^0(\mu) h(x - \mu) d\mu + \int_{-\infty}^{\infty} m_{\lambda_j}^0(\mu) h(x - \mu) d\mu + \int_{-\infty}^{\infty} r_{\lambda_j}^0(\mu) h(x - \mu) d\mu,
\]

where
\[
f_{\lambda_j}^0(\mu) = \begin{cases} 
f_{\lambda_j}^0(\mu) & \text{if } \mu \in (-\infty, -b_j) \\
0 & \text{if } \mu \not\in (-\infty, -b_j), \end{cases}
\]
\[
m_{\lambda_j}^0(\mu) = \begin{cases} 
f_{\lambda_j}^0(\mu) & \text{if } \mu \in [-b_j, b_j] \\
0 & \text{if } \mu \not\in [-b_j, b_j], \end{cases}
\]
\[
r_{\lambda_j}^0(\mu) = \begin{cases} 
f_{\lambda_j}^0(\mu) & \text{if } \mu \in (b_j, \infty) \\
0 & \text{if } \mu \not\in (b_j, \infty). \end{cases}
\]

\(^{25}\)Indeed, the arguments below could be extended to any “heterogeneity function” \(h\) that has limited support.
With regard to the first summand, \( \ell_{\lambda_j}(x) \), in the decomposition, we distinguish the following three regions for which we calculate the convolution integral separately: (a) \( x < -a - b_j \), (b) \(-a - b_j \leq x < a - b_j \), and (c) \( x \geq a - b_j \). This results in

\[
\ell_{\lambda_j}(x) = \begin{cases} 
\frac{1}{2a} \left( F^0_{\lambda_j}(x + a) - F^0_{\lambda_j}(x - a) \right) & \text{if } x < -a - b_j \\
\frac{1}{2a} \left( F^0_{\lambda_j}(-b_j) - F^0_{\lambda_j}(x - a) \right) & \text{if } a - b_j \leq x < a - b_j \\
0 & \text{if } x \geq a - b_j.
\end{cases}
\] (27)

Similarly, for the right part, \( r_{\lambda_j}(x) \), the limits of integration are \( x < -a + b_j \), \(-a + b_j \leq x < a + b_j \), and \( x \geq a + b_j \). By convolving we obtain

\[
r_{\lambda_j}(x) = \begin{cases} 
0 & \text{if } x < -a + b_j \\
\frac{1}{2a} \left( F^0_{\lambda_j}(x + a) - F^0_{\lambda_j}(b_j) \right) & \text{if } -a + b_j \leq x < a + b_j \\
\frac{1}{2a} \left( F^0_{\lambda_j}(x + a) - F^0_{\lambda_j}(x - a) \right) & \text{if } x \geq a + b_j.
\end{cases}
\] (28)

Finally, the convolution forming the middle part of the decomposition, \( m_{\lambda_j}(x) \), involves five regions of integration, viz. \( x < -a - b_j \), \(-a - b_j \leq x < -a + b_j \), \(-a + b_j \leq x < a - b_j \), \( a - b_j \leq x < a + b_j \), and \( x \geq a + b_j \). Combining results for these five regions, we have

\[
m_{\lambda_j}(x) = \begin{cases} 
0 & \text{if } x < -a - b_j \\
\frac{1}{2a} \left( F^0_{\lambda_j}(x + a) - F^0_{\lambda_j}(-b_j) \right) & \text{if } a - b_j \leq x < -a + b_j \\
\frac{1}{2a} \left( F^0_{\lambda_j}(b_j) - F^0_{\lambda_j}(-b_j) \right) & \text{if } -a + b_j \leq x \leq a - b_j \\
\frac{1}{2a} \left( F^0_{\lambda_j}(b_j) - F^0_{\lambda_j}(x - a) \right) & \text{if } a - b_j < x \leq a + b_j \\
0 & \text{if } x > a + b_j.
\end{cases}
\] (29)

Note that in the region where \(-a + b_j \leq x \leq a - b_j \), (29) is a constant function (for all \( j \)) which is due to the fact that the region of overlap between \( m^0_{\lambda_j} \) and \( h \) does not change in size as we slide \( h(x + \mu) \) over \( m^0_{\lambda_j}(\mu) \). Figure 7 depicts the density \( \hat{f}_{\lambda_j} \) of representative \( j \)'s ideal point as the sum of (27), (28), and (29). Then, \( \hat{f}_{\lambda_j}(x) \) can be approximated by

\[
\hat{f}_{\lambda_j}(x) = \begin{cases} 
\frac{1}{2a} \left( F^0_{\lambda_j}(b_j) - F^0_{\lambda_j}(-b_j) \right) & \text{if } a + b_j \leq x \leq a - b_j \\
0 & \text{if } x < -a + b_j \text{ or } x > a - b_j.
\end{cases}
\]

Conditional on the interval \([-a + b, a - b]\), the approximations \( \hat{f}_{\lambda_j} \) are identical for \( j = 1, \ldots, m \), i.e., the ideal points of representatives are approximately i.i.d.

The absolute approximation error we commit by using \( \hat{f}_{\lambda_j} \) rather than \( f_{\lambda_j}(x) \) is measured by the probability mass that we “discard”. This mass corresponds to the sum of the areas between \( \ell_{\lambda_j} \) and \( r_{\lambda_j} \) and the \( x \)-axis, respectively, and the areas between \( m_{\lambda_j} \) and the \( x \)-axis with \( x < -a + b_j \) and \( a - b_j \leq x \), respectively. It is equal to 1 less the area of a rectangle with width \( a - b_j - (-a + b_j) \) and height \( \frac{1}{2a} \left( F^0_{\lambda_j}(b_j) - F^0_{\lambda_j}(-b_j) \right) \) (see Figure 7). This gives formula (26).
This result implies that if \( \bar{b} \) is ‘small’ compared to \( a \), then \( \hat{f}_{\lambda_j} \) is a ‘good’ approximation of \( f_{\lambda_j} \) for every \( j \). If all \( n! \) orderings of ideal points had probability \( 1/n! \), the Shapley-Shubik linear rule would be correct. Conditional on the event that all ideal points fall into the interval \( [-a+\bar{b}, a-\bar{b}] \), the following approximation for the probability of any given ordering of representatives’ ideal points holds:

**Lemma 4.** Under the conditions of Lemma 3, the probability that any given ordering of representatives’ ideal points is realized is

\[
\frac{(a-\bar{b})^m}{a^m m!} \prod_{j=1}^{m} \left[ F_{\lambda_j}^0(b_j) - F_{\lambda_j}^0(-b_j) \right] + \rho(x_1, \ldots, x_m),
\]

where \( \vartheta_k \equiv \int_{-a+b}^{a-b} f_{\lambda_k}(x) dx, \ k = 1, \ldots, m \). The value of residual \( \rho(\cdot) \) depends on which particular profile one considers. In particular, if \( a \gg \bar{b} \) then \( \prod_{k} \vartheta_k \) and \( \prod_{j=1}^{m} \left[ F_{\lambda_j}^0(b_j) - F_{\lambda_j}^0(-b_j) \right] \) are close to 1, and

\[
\frac{(a-\bar{b})^m}{a^m m!} \prod_{j=1}^{m} \left[ F_{\lambda_j}^0(b_j) - F_{\lambda_j}^0(-b_j) \right] \approx \frac{1}{m!}.
\]

**Proof.** Let \( X_1, \ldots, X_m \) be independently distributed random variables with probability density functions \( f_1, \ldots, f_m \), respectively. Generally, the probability that the ordering \( x_1 < \cdots < x_m \) occurs is given by

\[
\int_{-\infty}^{x_m} \cdots \int_{-\infty}^{x_2} \int_{-\infty}^{x_1} f_m(x_m) \cdots f_1(x_1) \, dx_1 \cdots dx_m.
\]
If the \( f_j \) (\( j = 1, \ldots, m \)) are all identical, then expression (31) yields the obvious result that each possible ordering occurs with probability \( 1/m! \).

Consider the conditional probability density of representative \( j \)'s ideal point given that \( x \in [-a + \bar{b}, a - \bar{b}] \),

\[
 f_{\lambda_j}(x \mid [-a + \bar{b}, a - \bar{b}]) = \begin{cases} 
 \frac{f_{\lambda_j}(x)}{\int_{-a + \bar{b}}^{a + \bar{b}} f_{\lambda_j}(x)dx} & \text{if } -a + \bar{b} \leq x \leq a - \bar{b} \\
 0 & \text{if } x < -a + \bar{b} \text{ or } x > a - \bar{b}.
\end{cases}
\]

Let \( \vartheta_k \equiv \int_{-a + \bar{b}}^{a - \bar{b}} f_{\lambda_k}(x)dx \), \( k = 1, \ldots, m \). With this notation, the probability for the ordering \( x_1 < \cdots < x_m \) of representatives' ideal points conditional on the event that all ideal points fall into the interval \([-a + \bar{b}, a - \bar{b}]\) is given by

\[
 \frac{1}{\prod_k \vartheta_k} \int_{-a + \bar{b}}^{a - \bar{b}} \cdots \int_{-a + \bar{b}}^{a - \bar{b}} f_{\lambda_m}(x_m) \cdots f_{\lambda_1}(x_1) \, dx_1 \cdots dx_m. \tag{32}
\]

Using the decomposition of \( f_{\lambda_j}(x) \) defined above, (32) can be rewritten as

\[
 \frac{1}{\prod_k 2\vartheta_k} \int_{-a + \bar{b}}^{a - \bar{b}} \cdots \int_{-a + \bar{b}}^{a - \bar{b}} \prod_{j=1}^{m} \left( (F_{\lambda_j}^0(-b_j) - F_{\lambda_j}^0(x_j - a)) + (F_{\lambda_j}^0(b_j) - F_{\lambda_j}^0(-b_j)) + (F_{\lambda_j}^0(x_j + a) - F_{\lambda_j}^0(b_j)) \right) \, dx_1 \cdots dx_m. \tag{33}
\]

Carrying out the multiplication, one obtains

\[
 \frac{1}{\prod_k 2\vartheta_k} \int_{-a + \bar{b}}^{a - \bar{b}} \cdots \int_{-a + \bar{b}}^{a - \bar{b}} \prod_{j=1}^{m} \left( F_{\lambda_j}^0(b_j) - F_{\lambda_j}^0(-b_j) \right) \, dx_1 \cdots dx_m, \tag{34}
\]

where \( g(x_1, \ldots, x_m) \) is a residual capturing all product terms which depend on some \( x_j \).

The integral of \( g(x_1, \ldots, x_m) \) is specific to which particular profile one looks at, i.e., the order of integration is important here, while this is not the case for the integration of the product term in (34).

We now prove that

\[
 \int_{-a + \bar{b}}^{a - \bar{b}} \cdots \int_{-a + \bar{b}}^{a - \bar{b}} \, dx_1 \cdots dx_m = \frac{2^m}{m!} (a - \bar{b})^m. \tag{35}
\]

From this the lemma’s claim,

\[
 \frac{1}{\prod_k 2\vartheta_k} \int_{-a + \bar{b}}^{a - \bar{b}} \cdots \int_{-a + \bar{b}}^{a - \bar{b}} \prod_{j=1}^{m} \left( F_{\lambda_j}^0(b_j) - F_{\lambda_j}^0(-b_j) \right) \, dx_1 \cdots dx_m = \frac{(a - \bar{b})^m \prod_{j=1}^{m} \left( F_{\lambda_j}^0(b_j) - F_{\lambda_j}^0(-b_j) \right)}{a^m m! \prod_k \vartheta_k}, \tag{36}
\]
follows. Note that $F^0_\lambda(b_j) - F^0_\lambda(-b_j) \equiv \kappa$ for all $j$ by the definition of $b_j$.

To prove equation (35), first note that

$$\int_{-a+b}^{a-b} x_m \cdots \int_{-a-b}^{a+b} dx_1 \cdots dx_m = G_m(a - \bar{b}) - G_m(-a + \bar{b}),$$

where $G_m(t) \equiv \int_0^t \int_{a+b}^{a-m} \cdots \int_{a+b}^{a-m} dx_1 \cdots dx_m$ denotes the antiderivative after $m$ integrations.

**Claim 1:** For any positive integer $m$,

$$G_m(t) = \sum_{k=0}^{m-1} (-1)^k \frac{1}{(m-k)!} t^{m-k}(-a + \bar{b})^k. \quad (37)$$

**Proof by induction:** For $m = 1$, the above statement asserts that

$$\int_0^t dx_1 = G_m(t) = (-1)^0 \frac{1}{(1-0)!} t^{1-0}(-a + \bar{b})^0 = t$$

which is clearly true. Now assume that there is $m$ such that Claim 1 holds. We must now prove that the formula is true for $m+1$. By definition of $G_m(t)$, we have

$$G_{m+1}(t) = \int_0^t \left( G_m(x_{m+1}) - G_m(-a + \bar{b}) \right) dx_{m+1}.$$

Invoking the inductive assumption, one obtains

$$G_{m+1}(t) = \int_0^t \left( \sum_{k=0}^{m-1} (-1)^k \frac{1}{(m-k)!} x_{m+1}^{m-k}(-a + \bar{b})^k - \sum_{k=0}^{m-1} (-1)^k \frac{1}{(m-k)!} (-a + \bar{b})^m \right) dx_{m+1}$$

$$= \left[ \sum_{k=0}^{m-1} (-1)^k \frac{1}{(m+1-k)!} x_{m+1}^{m+1-k}(-a + \bar{b})^k - \sum_{k=0}^{m-1} (-1)^k \frac{1}{(m-k)!} (-a + \bar{b})^m x_{m+1} \right]_0^t$$

$$= \left[ \sum_{k=0}^{m} (-1)^k \frac{1}{(m+1-k)!} x_{m+1}^{m+1-k}(-a + \bar{b})^k + (-1)^m \frac{1}{(m-0)!} (-a + \bar{b})^m x_{m+1} \right]_0^t$$

$$= \sum_{k=0}^{m} (-1)^k \frac{1}{(m+1-k)!} x_{m+1}^{m+1-k}(-a + \bar{b})^k.$$
Step (*) follows by recognizing that \( \frac{1}{(m-k)k!} = \frac{1}{m} \binom{m}{k} \) and hence
\[
\sum_{k=0}^{m-1} (-1)^k \frac{1}{(m-k)!k!} (-a + \bar{b})^m x_{m+1} = \frac{1}{m!} (-a + \bar{b})^m x_{m+1} \sum_{k=0}^{m-1} (-1)^k \binom{m}{k}
\]
\[
= \frac{1}{m!} (-a + \bar{b})^m x_{m+1} \left( \sum_{k=0}^{m} (-1)^k \binom{m}{k} - (-1)^m \binom{m}{m} \right).
\]

Observing that
\[
\sum_{k=0}^{m} (-1)^k \binom{m}{k} = 0 \quad \text{for } m > 0 \quad (38)
\]
(see, e.g., Benjamin and Quinn 2003, p. 81), we arrive at the result given above. Thus, statement (37) indeed holds for \( m + 1 \).

Claim 2:
\[ G_m(a - \bar{b}) - G_m(-a + \bar{b}) = \frac{2^m}{m!} (a - \bar{b})^m \]

Proof: Using (37), we obtain that
\[
G_m(a - \bar{b}) - G_m(-a + \bar{b}) = \sum_{k=0}^{m-1} (-1)^k \frac{1}{(m-k)!k!} (-a + \bar{b})^{m-k} (-a + \bar{b}) - \sum_{k=0}^{m-1} (-1)^k \frac{1}{(m-k)!k!} (-a + \bar{b})^m
\]
\[
= (-a + \bar{b})^m \left( \sum_{k=0}^{m-1} (-1)^k \binom{m}{k} \frac{1}{m!} \left( \frac{m}{k} \right) - \sum_{k=0}^{m-1} (-1)^k \frac{1}{m!} \left( \binom{m}{k} \right) \right)
\]
\[
= (-a + \bar{b})^m \left( (-1)^m \sum_{k=0}^{m-1} \frac{1}{m!} \left( \binom{m}{k} \right) - \sum_{k=0}^{m-1} (-1)^k \frac{1}{m!} \left( \binom{m}{k} \right) + (-1)^m \frac{1}{m!} \left( \binom{m}{m} \right) \right).
\]

Then, equation (38) and the identity \( \sum_{k=0}^{m} \binom{m}{k} = 2^m \) imply that
\[
G_m(a - \bar{b}) - G_m(-a + \bar{b}) = (-a + \bar{b})^m \left( (-1)^m \sum_{k=0}^{m-1} \frac{1}{m!} \left( \binom{m}{k} \right) + (-1)^m \frac{1}{m!} \left( \binom{m}{m} \right) \right)
\]
\[
= (-a + \bar{b})^m \left( (-1)^m \frac{1}{m!} \sum_{k=0}^{m} \left( \binom{m}{k} \right) \right)
\]
\[
= (-1)^2 \frac{2^m}{m!} (a - \bar{b})^m.
\]

The implication of these two lemmas is

**Proposition 4.** If the relative error of approximation (30) is small, i.e., if
\[
\frac{(a - \bar{b})^m \prod_{j=1}^{m} \left[ F_{\lambda_j}(b_j) - F_{\lambda_j}(-b_j) \right]}{a^m \prod_k \vartheta_k} \approx 1,
\]

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then the weights \( w^* \) defined by

\[
SSI_j(q; w_1^*, \ldots, w_m^*) = \frac{1}{\sum_{k=1}^{m} n_k} \sum_{k=1}^{n_j} n_k, \quad j = 1, \ldots, m,
\]

approximately satisfy the egalitarian criterion (3) for any given decision quota \( q \).

References


